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RADIO WAVE SCATTERING FROM RANDOM FLUCTUATIONS IN AN ANISOTROPIC MEDIUM

BY

Dale M. Simonich
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Ionosphere Radio Laboratory

Electrical Engineering Research Laboratory
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ABSTRACT

This work treats the problem of scattering of electromagnetic waves from random electron density fluctuations in a cold magneto-plasma. The problem is formulated in terms of characteristic waves. First a Green's function is developed as a superposition of charzcteristic waves. This Green's function is asymptotic for large distances and thus the solutions are for far fields only. This Green's function is then used to solve two kinds of problems using the Born or single scatter approximation. The first problem considered is the bistatic scattering problem. Here a transmitter beams power at a region in space which contains random fluctuations in electron density. An expression is obtained for the power scattered in a given direction from the volume. The results are then discussed. The second problem is the radio star problem. In this problem a radio wave propagates through an infinite slab and is received below the slab. Expressions for the correlations of the in phase and quadriture phase components of the electric field along the unscattered ray path (longitudinal correlation) are obtained. These expressions are then discussed.

There are two general conclusions that can be made from this work. The first is that the solution can depend very strongly on the relationship among the incident ray vector,

the scattered ray vector and the D.C. magnetic field vector. The degree of variation increases as the medium becomes more anisotropic. The second conclusion is that mode conversion is caused by those fluctuations whose correlation length is of the order of the wavelength. If the correlation length is much larger than the wavelength mode conversion is negligible.

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CHAPTER I

INTRODUCTION

1.1 Description of the Problem

The problem of the scattering of electromagnetic fields in a random medium when the background medium is isotropic has been discussed extensively in the literature, but not much work has been done when the background medium is anisotropic. The purpose of the work presented here is to extend the work in the area of anisotropic background media.

The anisotropic random medium to be considered here is a magneto-plasma with random fluctuations in the number density of electrons. The frequencies considered are high enough so that the effect of ions can be ignored and therefore as far as the electromagnetic field is concerned only electrons need be considered important. For this plasma two things are of interest. Since the background medium is anisotropic its properties depend upon direction and therefore one area of interest is the variation of the solution with respect to variations in the geometry of the problem, including the D.C. magnetic field vector. The second area of interest is the depolarization of the signal, that is, the generation of electromagnetic waves with a different polarization than the incident wave.

Two different types of geometries are usually investigated in problems of this type. The first is the bistatic geometry. In this problem energy originates at a transmitter and propagates into a region containing random fluctuations which scatter some of the energy. This scattered energy is then detected by a receiver. The volume from which the energy is scattered is assumed to be small. Either the scattering volume itself is small or the beams of the transmitting and receiving antennas are narrow so that their common volume is small. This type of geometry occurs in the study of laboratory plasmas using microwave beams and in the study of the ionosphere by ionospheric sounders, partial reflection techniques and oblique sounding techniques.

The second type of geometry is the so-called radio star problem. In this problem the beams of the transmitter and receiver are pointed at each other. Between them is a slab which contains random fluctuations. There is no attempt to restrict the size of the scattering region transverse to the path joining the transmitter and receiver as there was in the previous case. This type of geometry occurs in the reception of signals from radio stars and earth satellites as well as certain laboratory experiments.

1.2 History

The early work in this area was done by Bergman (1946) and Pekeris (1947). Booker and Gordon (1950) used the method of Perkeris to solve the bistatic problem for an isotropic medium with isotropic irregularities. This work was then generalized to non-isotropic irregularities by Booker (1956). The results of this work shows that the scattering cross section is given by

$$\sigma = \frac{<\Delta\varepsilon^2>}{\varepsilon^2} \frac{\pi^2 \sin^2\chi}{\lambda^4} \; P[k(\hat{r}_2 - \hat{r}_1)]$$

where χ is the angle between the direction of scattering $\hat{r}_2$ and the direction of the incident electric field $\vec{E}_0$, $\hat{r}_1$ is the direction of the transmitter and P is the spectrum of $\Delta\varepsilon$. The book by Tatarski (1961) presents a very detailed discussion of this problem and is therefore a very good reference. Cohen (1962) considered the problem of scattering in a warm isotropic plasma. In addition to the scattering cross sections of the electromagnetic wave and plasma wave he developed conversion cross sections between them.

The early work on the radio star problem can be found in the review paper of Ratcliffe (1956). The extensive bibliography in this paper provides a good survey of the early literature. The background medium was isotropic and the problem was formulated in terms of diffracting screens or phase and amplitude modulating screens. Bowhill (1961) used this same technique to consider strong fluctuations. Budden (1965) considered the problem of correlation of amplitude fluctuations expressing the result in terms of the spectral intensity function. Tatarski (1961) also used this type of approach. The book by Chernov (1960) presents a comprehensive treatment of the correlations of the log amplitude and phase departure for isotropic irregularities. Yeh (1962) considered the problem of scattering in an isotropic medium with non-isotropic fluctuations. The method of solution used here corresponds to the treatment in these last two works. Yeh and Liu (1967) considered the problem with an anisotropic background medium but took it only to the first order in Y and therefore considered only high frequencies. Their interest was in the Faraday effect.

The method used to obtain the asymptotic Green's function was first presented by Lighthill (1960). It has been used by other authors such as Felson (1965) and Deschamps (1964). Kesler (1965) presents a very comprehensive development of the use of characteristic wave expansions as well as the asymptotic solution used here.

1.3 Description of the Text

Chapter two is devoted to the determination of the Green's function and a discussion of the pertinent factors in it. First the equation for the scattered field is developed. Next the characteristic waves are normalized and an expression for the Green's function as a superposition of characteristic waves is obtained. An asymptotic evaluation of this Green's function is then obtained. Finally the characteristic waves and the Gaussian curvature are discussed.

Chapter three is devoted to the bistatic scattering problem. First an expression for the scattered power is obtained and then the scattering cross section is developed. The solution divides nicely into a part which depends upon the properties of the medium and the geometry of the problem and a part which in addition depends upon the statistics of the fluctuations. These two factors are discussed in separate sections. The approximations that were made in developing the solution are discussed next. Finally two applications of the theory are presented.

Chapter four is devoted to the radio star problem. First expressions for the longitudinal correlations of the in phase and quadriture phase components of the scattered field are obtained. The fluctuations are assumed to be isotropically correlated. The resulting equations are then discussed. Finally the approximations used in the solution are discussed.

Chapter five is the concluding chapter. The important points developed are emphasized and the limitations of the work are discussed. Some comments on future work are also presented.

CHAPTER II

THE GREEN'S DYADIC $\overline{\overline{\Gamma}}(\vec{r}|\vec{\bar{r}}\,')$ AS A SUPERPOSITION OF CHARACTERISTIC WAVES

2.1 Introduction

This chapter is devoted to the derivation of the differential equation for the scattered field in an anisotropic medium and the asymptotic solution for the Green's function which will be used to solve for the two general classes of problems considered in Chapters III and IV. Some characteristics of cold magneto-plasmas are also presented insofar as they are involved in the Green's function.

2.2 The Scattered Field

The general scattering problem consists of an incident wave propagating in the medium which encounters a region of space in which there are random fluctuations in number density of the electrons and thus random fluctuations in the relative dielectric tensor $\overline{\overline{K}}$. The object is to find an expression for the scattered field. In the work presented here the single scatter or Born approximation will be used.

In the scattering region V the incident electric field E_i and the scattered electric field E_s satisfy the equation

$$\nabla x \nabla x[\vec{E}_i(\vec{r})+\vec{E}_s(\vec{r})] - k_0^2\overline{\overline{K}}(\vec{r})\cdot[\vec{E}_i(\vec{r})+\vec{E}_s(\vec{r})] = 0 \qquad 2.2.1$$

where k_0^2 is the free space wave number, $k_0 = \omega/C$. Let the relative dielectric tensor $\overline{\overline{K}}(\vec{r})$ be given by

$$\overline{\overline{K}}(\vec{r}) = \overline{\overline{I}} - X\overline{\overline{M}} - \Delta X(\vec{r})\overline{\overline{M}}$$

where $\overline{\overline{I}}$ is the unit tensor,

$$\overline{\overline{M}} = \frac{1}{1-Y^2}\begin{bmatrix} 1-Y_x^2 & -Y_xY_y+jY_z & -Y_xY_z-jY_y \\ -Y_xY_y-jY_z & 1-Y_y^2 & -Y_yY_z+jY_x \\ -Y_xY_z+jY_y & -Y_yY_z-jY_x & 1-Y_z^2 \end{bmatrix} \quad 2.2.2$$

$\vec{Y} = \frac{e}{m\omega}\vec{B}_0$ for electrons, $\vec{B}_0$ is the D. C. magnetic field, ω is the angular operating frequency, $X = \frac{Ne^2}{m\varepsilon_0}$, N is the number density and $\Delta X(\vec{r})$ is the random fluctuation in X. Using this form for $\overline{\overline{K}}$, equation 2.2.1 becomes

$$\nabla x \nabla x[\vec{E}_i(\vec{r})+\vec{E}_s(\vec{r})] - k_0^2[\overline{\overline{I}}-X\overline{\overline{M}}-\Delta X(\vec{r})\overline{\overline{M}}]\cdot[\vec{E}_i(\vec{r})+\vec{E}_s(\vec{r})] = 0 \quad 2.2.3$$

Now if $\Delta X = 0$, there is no scattered field and $\vec{E}_i$ satisfies the equation

$$\nabla x \nabla x \vec{E}_i(\vec{r}) - k_0^2(\overline{\overline{I}}-X\overline{\overline{M}})\cdot\vec{E}_i(\vec{r}) = 0$$

Thus equation 2.2.3 becomes

$$\nabla x \nabla x \vec{E}_s(\vec{r}) - k_0^2(\overline{\overline{I}}-X\overline{\overline{M}})\cdot\vec{E}_s(\vec{r}) = -k_0^2\Delta X(\vec{r})\overline{\overline{M}}\cdot[\vec{E}_i(\vec{r})+\vec{E}_s(\vec{r})] \quad 2.2.4$$

This is an equation for the scattered field $\vec{E}_s$ with the random fluctuations as the source function. Unfortunately this source function depends upon the scattered field, and therefore some approximation is desirable. In the Born approximation the scattered field is assumed to be so small that the electric field at the scatterer is essentially the incident field $\vec{E}_i$ ($E_i >> E_s$). This approximation ignores the contribution of the scattered field to the source term and is therefore referred to as the single scatter approximation. Therefore equation 2.2.4 becomes

$$\nabla x \nabla x \vec{E}_s(\vec{r}) - k_0^2(\overline{\overline{I}} - X\overline{\overline{M}}) \cdot \vec{E}_s(\vec{r}) = -k_0^2 \Delta X(\vec{r})\overline{\overline{M}} \cdot \vec{E}_i \qquad 2.2.5$$

This is the differential equation for the scattered field which is used in this paper. The solution of this equation can be formally written as

$$\vec{E}_s(\vec{r}) = -k_0^2 \int_V \Delta X(\vec{r}\,')\overline{\overline{\Gamma}}(\vec{r}|\vec{r}\,') \cdot \overline{\overline{M}} \cdot \vec{E}_i(\vec{r}\,')d\vec{r}\,' \qquad 2.2.6$$

where V is the scattering volume and $\overline{\overline{\Gamma}}(\vec{r}|\vec{r}\,')$ is the Green's dyadic for the operator $[\nabla x \nabla x - k_0^2(\overline{\overline{I}} - X\overline{\overline{M}}) \cdot]$. In the following sections an asymptotic solution for $\overline{\overline{\Gamma}}$ in terms of characteristic waves will be found.

2.3 Normalization of the Characteristic Waves

Consider a cold magneto-plasma described by the relative dielectric tensor $\overline{\overline{K}} = \overline{\overline{I}} - X\overline{\overline{M}}$. The solution of the eigen-value problem for electromagnetic waves in this medium yields two characteristic waves or normal modes (Ratcliffe, 1959, Chapter 2), which are electromagnetic in nature. A characteristic wave is a wave which does not change its state of polarization as it propagates in a given direction. In general the properties of a characteristic wave will be a function of the direction of propagation.

Consider the wave equation

$$\nabla x \nabla x \vec{E}(\vec{r}) - k_0^2 \overline{\overline{K}} \cdot \vec{E}(\vec{r}) = 0 \qquad 2.3.1$$

The Fourier transform of this equation is

$$(k^2 \overline{\overline{I}} - \vec{k}\vec{k} - k_0^2 \overline{\overline{K}}) \cdot \vec{E}(\vec{k}) = 0 \qquad 2.3.2$$

where $\vec{E}(\vec{k}) = \int_{-\infty}^{\infty} \vec{E}(\vec{r})\, e^{j\vec{k}\cdot\vec{r}} d\vec{r}$. As was stated above there are two characteristic waves which satisfy this equation. They are given by

$$(k_m^2 \overline{\overline{I}} - \vec{k}_m \vec{k}_m - k_0^2 \overline{\overline{K}}) \cdot \vec{a}_m = 0 \qquad 2.3.3a$$

$$(k_n^2 \overline{\overline{I}} - \vec{k}_n \vec{k}_n - k_0^2 \overline{\overline{K}}) \cdot \vec{a}_n = 0 \qquad 2.3.3b$$

where $\vec{k}_m$ is the wave vector of the characteristic wave m and $\vec{a}_m$ is the polarization vector, similarly for $\vec{k}_n$ and $\vec{a}_n$.

Now multiply equation 2.3.3a by $\vec{a}_n^*$ from the left and equation 2.3.3b by $\vec{a}_m^*$ from the left to obtain

$$\vec{a}_n^* \cdot (k_m^2 \overline{\overline{I}} - \vec{k}_m \vec{k}_m - k_0^2 \overline{\overline{K}}) \cdot \vec{a}_m = 0 \qquad 2.3.4a$$

$$\vec{a}_m^* \cdot (k_n^2 \overline{\overline{I}} - \vec{k}_n \vec{k}_n - k_0^2 \overline{\overline{K}}) \cdot \vec{a}_n = 0 \qquad 2.3.4b$$

If the medium described by $\overline{\overline{K}}$ is lossless, $\overline{\overline{K}}$ is hermitian ($\overline{\overline{K}}^+ = \overline{\overline{K}}$) (Landau and Lifshitz, 1960, Section 82). If $\overline{\overline{K}}$ is hermitian, the wave vectors $\vec{k}_m$ and $\vec{k}_n$ are real in the propagating region (Hoffman and Kunze, 1961, Section 8.6).

Taking the hermitian conjugate of equation 2.3.4b and using the above properties of $\overline{\overline{K}}$ and $\vec{k}_m$ a new equation 2.3.4b' can be derived

$$\vec{a}_n^* \cdot (k_n^2\overline{\overline{I}}-\vec{k}_n\vec{k}_n-k_0^2\overline{\overline{K}}) \cdot \vec{a}_m = 0 \qquad 2.3.4b'$$

Subtract equation 2.3.4b' from equation 2.3.4a to obtain

$$\vec{a}_n^* \cdot [(k_m^2-k_n^2)\overline{\overline{I}}-(\vec{k}_m\vec{k}_m-\vec{k}_n\vec{k}_n)] \cdot \vec{a}_m = 0 \qquad 2.3.5$$

Since the characteristic waves are defined for a given direction of propagation, let $\vec{k}_m = \vec{k}_m\ \hat{p}$ and $\vec{k}_n = \vec{k}_n\ \hat{p}$, where $\hat{p}$ is a unit vector in the direction of propagation. With this substitution equation 2.3.5 becomes

$$(k_m^2-k_n^2)\vec{a}_n^* \cdot (\overline{\overline{I}}-\hat{p}\hat{p}) \cdot \vec{a}_m = 0 \qquad 2.3.6$$

There are two possible solutions to equation 2.3.6 which give the orthogonality condition and the normalization for the polarization vector $\vec{a}$. If $m \neq n$, then in general the first factor is non-zero, so

$$\vec{a}_n^* \cdot (\overline{\overline{I}}-\hat{p}\hat{p}) \cdot \vec{a}_m = 0 \qquad 2.3.7$$

which is part of the orthogonality relation. If $m = n$ the first factor is zero so the second factor can be chosen arbitrarily. Normalize the polarization vector so that

$$\vec{a}_m^* \cdot (\overline{\overline{I}}-\hat{p}\hat{p}) \cdot \vec{a}_m = 1 \qquad 2.3.8$$

The meaning of this normalization relation can be examined simply in a special coordinate system in which $\hat{p} = \hat{z}$. Then

$$\bar{\bar{I}} - \hat{p}\hat{p} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using this in equation 2.3.8 shows that the characteristic waves are normalized such that the magnitude of the polarization vector transverse to the propagation vector is equal to unity.

Equations 2.3.7 and 2.3.8 can be condensed and restated as the orthonormal condition,

$$\vec{a}_n^* \cdot (\bar{\bar{I}}-\hat{p}\hat{p}) \cdot \vec{a}_m = \delta_{mn} \qquad 2.3.9$$

where δ_{mn} is the Kronecker delta. Another relationship can be obtained from equation 2.3.4b' using equation 2.3.9. It is the orthogonality relation

$$\vec{a}_n^* \cdot k_0^2\bar{\bar{K}} \cdot \vec{a}_m = k_n^2\ \delta_{mn} \qquad 2.3.10$$

Equations 2.3.9 and 2.3.10 give the orthonormal relationships for the polarization vector $\vec{a}$ which are needed to determine a solution for the Green's function $\bar{\bar{\Gamma}}$ as a superposition of characteristic waves.

2.4 The Green's Dyadic $\overline{\overline{\Gamma}}\,(\vec{r}|\vec{r}\,')$

The Green's dyadic $\overline{\overline{\Gamma}}\,(\vec{r}|\vec{r}\,')$ will be determined by solving the equation for the scattered electric field as a superposition of characteristic waves. The resulting equation will be compared to the formal solution for $\vec{E}_s$ by the method of Green's functions resulting in the identification of the Green's dyadic.

In the infinite medium described by the relative dielectric tensor $\overline{\overline{K}} = \overline{\overline{I}} - X\overline{\overline{M}}$, the electric field must satisfy the equation

$$\nabla\times\nabla\times\,\vec{E}(\vec{r}) - k_0^2\overline{\overline{K}}\cdot\vec{E}(\vec{r}) = \vec{J}(\vec{r}) \qquad 2.4.1$$

where $\vec{J}(\vec{r})$ is some given current density. Take the Fourier transform of this equation

$$(k^2\overline{\overline{I}} - \vec{k}\vec{k} - k_0^2\overline{\overline{K}})\cdot\vec{E}(\vec{k}) = \vec{J}(\vec{k}) \qquad 2.4.2$$

where $\vec{k}$ is the transform variable defined in section 2.3. Since the characteristic waves form a basis for the solutions to the above equation (Collin, 1960; appendix A.3), the solution for $\vec{E}(\vec{k})$ can be expressed as

$$\vec{E}(\vec{k}) = \sum_m A_m\,\vec{a}_m \qquad 2.4.3$$

where A_m is an amplitude and $\vec{a}_m$ is a normalized characteristic polarization corresponding to the direction $\hat{k}$. Using this expansion equation 2.4.2 can be written as

$$\sum_m A_m (k^2\overline{\overline{I}} - \vec{k}\vec{k} - k_0^2\overline{\overline{K}}) \cdot \vec{a}_m = \vec{J}(\vec{k}) \qquad 2.4.4$$

Take the scalar product of equation 2.4.4 with $\vec{a}_n^*$.

$$\sum_m A_m \vec{a}_n^* \cdot (k^2\overline{\overline{I}} - \vec{k}\vec{k} - k_0^2\overline{\overline{K}}) \cdot \vec{a}_m = \vec{a}_n^* \cdot \vec{J}(\vec{k})$$

Let the vector $\hat{k}$ be in the propagation direction for the characteristic wave defined by $\vec{a}$, the orthogonality relations given by equations 2.3.9 and 2.3.10 can be used to determine the amplitude A_m of characteristic wave m as

$$A_m = \frac{\vec{a}_m^*}{k^2 - k_m^2} \cdot \vec{J}(\vec{k}) \qquad 2.4.5$$

Thus from equation 2.4.3, the electric field is

$$\vec{E}(\vec{k}) = \sum_m \frac{\vec{a}_m^* \cdot \vec{J}(\vec{k})}{k^2 - k_m^2} \vec{a}_m \qquad 2.4.6$$

Now take the inverse transform of this equation

$$\vec{E}(\vec{r}) = \frac{1}{(2\pi)^3} \int \sum_m \frac{\vec{a}_m \vec{a}_m^*}{k^2 - k_m^2} \cdot \vec{J}(\vec{k})\, e^{-j\vec{k}\cdot\vec{r}}\, d\vec{k} \qquad 2.4.7$$

Substituting $\vec{J}(\vec{k}) = \int J(\vec{r}\,')e^{j\vec{k}\cdot\vec{r}\,'}d\vec{r}\,'$ into this equation 2.4.7 and rearranging terms, this equation becomes finally,

$$\vec{E}(\vec{r}) = \int_V \left[\frac{1}{(2\pi)^3} \sum_m \int \frac{\vec{a}_m \vec{a}_m^*}{k^2 - k_m^2} e^{-j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, d\vec{k}\right] \cdot \vec{J}(\vec{r}\,')d\vec{r}\,' \qquad 2.4.8$$

where the volume V is the support of $\vec{J}(\vec{r})$.

The solution to equation 2.4.1 by the method of Green's functions can be formally written as

$$\vec{E}(\vec{r}) = \int_V \overline{\overline{\Gamma}}\,(\vec{r}|\vec{r}\,') \cdot \vec{J}(\vec{r}\,')d\vec{r}\,' \qquad 2.4.9$$

By comparing equations 2.4.8 and 2.4.9 the Green's dyadic can be identified as

$$\overline{\overline{\Gamma}}(\vec{r}|\vec{r}\,') = \frac{1}{(2\pi)^3} \sum_m \int \frac{\vec{a}_m \vec{a}_m^{\,*}}{k^2-k_m^2} e^{-j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, d\vec{k} \qquad 2.4.10$$

While this equation gives a formal expression for the Green's dyadic, the integrations involved in obtaining an analytic expression for $\overline{\overline{\Gamma}}$ are very difficult if not impossible. For the cases to be considered here no exact solution to equation 2.4.10 has ever been found. In the next section an approximation to equation 2.4.10 for large distances from the source region will be obtained.

2.5 Asymptotic Evaluation of the Green's Dyadic

The method used for the asymptotic evaluation of equation 2.4.10 was first presented by Lighthill (1960). The method has been used by others in the field of wave propagation; for example Felson (1964) and Deschamps (1964). It gives an asymptotic solution for large $\vec{R}$ which is $0(\frac{1}{R^2})$. Since the type of problems to which the results are to be applied usually desire the solution a large distance from the scattering region, this restriction is not too great.

The general class of problem for which a solution is sought is

$$\overline{\overline{I}}_m(\vec{R}) = \int_{-\infty}^{\infty} \frac{\overline{\overline{A}}(\vec{k}_m)}{k^2-k_m^2} e^{-j\vec{k}\cdot\vec{R}} d\vec{k} \qquad 2.5.1$$

where $\vec{k}$ and $\vec{k}_m$ are in the same direction ($\vec{k} = k\,\hat{k}$ and $\vec{k}_m = k_m\,\hat{k}$). The solution is then given as the result of two different methods of integration. The first is the use of contour integration and the theory of residues to reduce the integral to a two fold integral. If $\overline{\overline{A}}(\vec{k}_m)$ is regular, the pole of the integrand occurs at $k = k_m$. This first integration then shows that the transform variable $\vec{k}$ must be identical to the wave normal $\vec{k}_m$. The remaining two integrals are then solved by an asymptotic technique. The method chosen to solve these remaining integrals is the method of stationary phase.

The method of stationary phase makes use of the fact that a large constant R appears in a phase term along with variables of integration. This will cause the factor $e^{-j\vec{k}_m\cdot\vec{R}}$ to oscillate very rapidly except near those points where $\vec{k}_m$ is stationary. If the rest of the integrand is slowly varying compared to this factor, these rapid oscillating sections will give almost zero contribution to the integral. Thus the main contribution to the value of the integral will come from the regions around the stationary points of $\vec{k}_m$. The stationary points are those points for which the exponent has zero first derivative with respect to the variables of integration. This condition implies that the normal to the surface defined by $\vec{k}_m$ is in the direction of the ray vector $\vec{R}$. The details of this work can be found in Lighthill (1960), only the results will be quoted.

The solution to equation 2.5.1 is given by equation 74 of Lighthill's (1960) paper. It is

$$\overline{\overline{I}}_m(\vec{R}) \approx \frac{4\pi^2}{R} \sum_s \frac{d_{ms}\ \overline{\overline{A}}(k_{ms})}{|\nabla_k(k^2-k_m^2)|_s \sqrt{|C_{ms}|}}\ e^{-j\ \vec{k}_{ms}\cdot\vec{R}} \qquad 2.5.2$$

where m refers to a characteristic mode and s refers to a stationary point for that mode. The summation over s takes into account the possibility of more than one point satisfying the condition of stationary phase, which are those points of the surface $k = k_m$ for which the normal is parallel to $\vec{R}$. d_{ms} is a constant that is (a) $\pm j$ if $C_{ms} < 0$ and $\nabla_k(k^2-k_m^2)|_s$ is in the direction $\pm \vec{R}$, (b) ± 1 if $C_{ms} > 0$ and the surface is convex to $\pm \nabla_k(k^2-k_m^2)|_s$. C_{ms} is the Gaussian curvature of the surface evaluated at the stationary point.

Equation 2.5.2 is valid if $C_{ms} \neq 0$. If this is not the case, solutions are still possible for some cases, see Lighthill (1960, section 5). For the type of surfaces considered here most cases of vanishing Gaussian curvature lead to solutions in terms of the Airy integral. The interesting feature of this is that when $C = 0$ the fields decay like $R^{-5/6}$ instead of R^{-1}. This means that regions of vanishing Gaussian curvature have fields associated with them which are decaying slower with distance.

Consider now the factor $|\nabla_k (k^2-k_m^2)|_s$. The derivative of $(k^2-k_m^2)$ in the direction $\hat{k}$ is

$$\frac{d}{dk}(k^2-k_m^2)|_s = 2k_{ms}$$

since $\vec{k}_m$ and $\vec{k}$ have only their direction in common originally. But

$$\frac{d}{dk}(k^2-k_m^2) = \nabla_k (k^2-k_m^2) \cdot \hat{k}$$

$$= |\nabla_k (k^2-k_m^2)| \cos \alpha_m$$

where α_m is the angle between $\nabla_k (k^2-k_m^2)$ and $\vec{k}$. The stationary phase condition requires that $\nabla_k (k^2-k_m^2)$ evaluated at the stationary point be in the direction $\vec{R}$. Therefore $\cos \alpha_{ms} = \hat{k}_{ms} \cdot \hat{R}$. Thus

$$|\nabla_k (k^2-k_m^2)|_s = 2k_{ms} \sec \alpha_{ms}$$

Equation 2.5.2 can now be written as

$$\overline{\overline{I}}_m(\vec{R}) = \frac{2\pi^2}{R} \sum_s \frac{d_{ms}\, \overline{\overline{A}}(\vec{k}_{ms})}{k_{ms} \sec \alpha_{ms} \sqrt{|C_{ms}|}} e^{-j\, \vec{k}_{ms} \cdot \vec{R}} \qquad 2.5.3$$

This result can be used to solve equation 2.4.10. The result is

$$\overline{\overline{\Gamma}}(\vec{r}\, \vec{r}\,') = \frac{1}{4\pi R} \sum_s \frac{d_s\, \vec{a}(\vec{k}_s)\vec{a}^*(\vec{k}_s)}{k_s \sec \alpha_s \sqrt{|C_s|}} e^{-j\, \vec{k}_s \cdot \vec{R}} \qquad 2.5.4$$

where $\vec{R} - \vec{r} - \vec{r}\,'$ and the notation ms, used to designate a stationary point has been dropped in favor of a single index s for simplicity. In the rest of the text s = 1 will refer to the ordinary mode for which there is only one stationary point, $s = 2 - 4$ will refer to the extraordinary mode for which there may be as many as three stationary points.

Since the normal to the surface defined by $\vec{k}_s$ must be in the direction of $\vec{R}$, $\vec{k}_s$ is a function of $\vec{R}$ and because of this d_s, $\vec{a}_s$, α_s and C_s will also be functions of $\vec{R}$. Therefore equation 2.5.4 is a very complicated function of $\vec{R}$ in general.

The next two sections will discuss the quantities in equation 2.5.4 for a cold plasma.

2.6 Characteristic Waves

The medium to which this paper will be applied is a cold magneto-plasma. A cold magneto-plasma is one in which the pressure term in the equation of motion is ignored. This is justified if the thermal velocity of the electrons is much less than the phase velocity of the wave, hence the name cold.

The characteristic waves for a medium described by $\overline{\overline{K}}$ satisfy

$$(k^2\overline{\overline{I}} - \vec{k}\,\vec{k} - k_0^2\,\overline{\overline{K}})\cdot\vec{a} = 0 \qquad 2.6.1$$

There is a non-trivial solution to equation 2.6.1 if

$$\det\,(k^2\overline{\overline{I}} - \vec{k}\,\vec{k} - k_0^2\,\overline{\overline{K}}) = 0 \qquad 2.6.2$$

which leads to the refractive index ($k = k_0 n$)

$$n^2 = 1 - \frac{X}{1 - \frac{Y^2 \sin^2\theta}{2(1-X)} \pm \sqrt{\frac{Y^4 \sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta}} \qquad 2.6.3$$

called the Appleton-Hartree equation (Ratcliffe, 1959, section 2.5). X and Y have been previously defined in section 2.2, and θ is the angle between the wave normal $\vec{k}$ and the D. C. magnetic field. In this equation Y is positive for electrons. This equation is valid when the frequency ω is much higher than any ion associated frequencies so that only the electrons are considered movable. In this equation the + sign refers to the so called ordinary wave and the - sign refers to the extra-ordinary wave.

In the coordinate system shown in Figure 2.6.1 $\vec{a}$ can

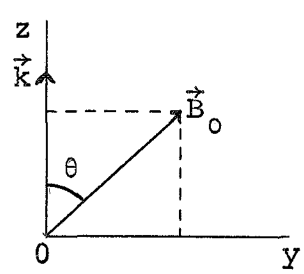


Figure 2.6.1 Special Coordinate System for $\vec{a}$

be written as (Ratcliffe, 1959, section 2.5)

$$\vec{a} = \frac{1}{\sqrt{1+|R|^2}} [R, 1, RQ] \qquad 2.6.4a$$

where

$$R = \frac{j}{\cos\theta}\left[\frac{Y\sin^2\theta}{2(1-X)} \pm \sqrt{\frac{Y^2\sin^4\theta}{4(1-X)^2} + \cos^2\theta}\right] \qquad 2.6.4b$$

$$Q = \frac{j\,Y\sin\theta}{1-X}(1-n^2) \qquad 2.6.4c$$

The normalization of equation 2.3.8 has been used to get equation 2.6.4a.

The results of the previous section require $\vec{k}_s$ and $\vec{a}_s$ for the ray direction $\vec{R}$. The following procedure is used to find them:

1. If ρ is the angle $\vec{R}$ makes with $\vec{B}_0$, called the ray angle, the stationary phase condition gives values of θ for which the normal to the surface makes an angle ρ with $\hat{B}_0$.
2. Using this value of θ in equations 2.6.3 and 2.6.4 $\vec{k}_s$ and $\vec{a}_s$ in the coordinate system of Figure 2.6.1 are obtained ($\vec{k}_s = k_s\,\hat{z}$).
3. $\vec{k}_s$ and $\vec{a}_s$ are then transformed into the coordinate system of the particular problem being solved.

Step one is the most difficult since there is no explicit solution for θ as a function of ρ. Therefore for a given X and Y a table is generated which gives corresponding values of ρ and θ. Then for a specific ray direction this table is scanned to find the value(s) of θ which correspond to it.

2.7 The Gaussian Curvature C

In a magneto-ionic medium the D. C. magnetic field is an axis of symmetry. The surface defined by $\vec{k}_s$ is therefore a surface of revolution about the D. C. magnetic field. For this case the determination of the Gaussian curvature becomes much simpler.

The Gaussian curvature for a surface of revolution is given by O'Neill (1966) as

$$C(u) = - \frac{g'(u)\ [g'(u)\ h''(u) - h'(u)\ g''(u)]}{h(u)\ [g'^2(u) + h'^2(u)]^2} \qquad 2.7.1$$

where the prime denotes differentiation and g and h are defined in Figure 2.7.1.

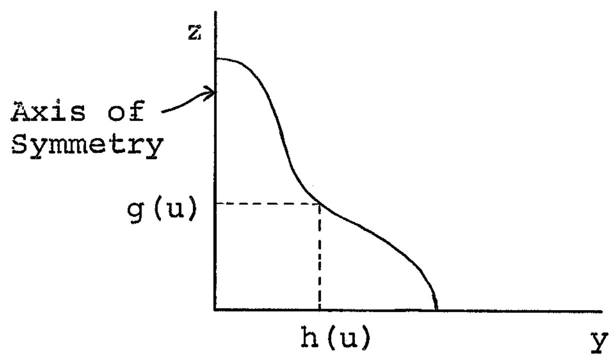


Figure 2.7.1 The Definition of g(u) and h(u)

For a magneto-ionic medium these parameters are $g(\theta) = k(\theta) \cos \theta$ and $h(\theta) = k(\theta) \sin \theta$. Using these functions in equation 2.7.1 the Gaussian curvature becomes

$$C(\theta) = \left(1 - \frac{k'(\theta)}{k(\theta)} \cot \theta\right) \frac{k^2(\theta) + 2k'^2(\theta) - k(\theta)k''(\theta)}{[k^2(\theta) - k'^2(\theta)]^2} \qquad 2.7.2$$

This equation can be simplified by using the stationary phase condition. Since the phase term must be stationary, $\nabla_k \vec{k}_m \cdot \hat{R} = 0$. With respect to the D. C. magnetic field $\vec{k}$ makes an angle θ and $\hat{R}$ makes an angle ρ therefore $\nabla_k k_m \cos(\rho-\theta) = 0$. But $k_m = k_m(\theta)$ so this equation reduces to $\frac{d}{d\theta}[k_m(\theta) \cos(\rho-\theta)] = 0$. This leads to the result that

$$\tan \alpha = -\frac{1}{k(\theta)} \frac{dk(\theta)}{d\theta} \qquad 2.7.3$$

where $\alpha = \rho - \theta$. Using this result equation 2.7.2 becomes

$$C(\theta) = \frac{\sin \rho \cos \alpha}{\sin \theta} (1 + \sin^2 \alpha - \frac{k''(\theta)}{k(\theta)} \cos^2 \alpha) \frac{1}{k^2(\theta)} \qquad 2.7.4$$

For a given ray direction ρ and a given refractive index surface n the value of θ is determined by equation 2.7.3

Equation 2.7.4 can be written in a different form. Take the derivative of equation 2.7.3 with respect to θ. After a little algebra

$$\frac{d\rho}{d\theta} = 1 + \sin^2 \alpha - \frac{k''}{k} \cos^2 \alpha \qquad 2.7.5$$

Therefore equation 2.7.4 becomes

$$C(\theta) = \frac{\sin \rho \cos \alpha}{\sin \theta} \frac{d\rho}{d\theta} \frac{1}{k^2(\theta)} \qquad 2.7.6$$

CHAPTER III

THE BISTATIC SCATTERING PROBLEM

3.1 Introduction

The bistatic scattering problem has a geometry which is the same as a two-point communication problem. An electromagnetic wave is transmitted in a narrow beam at a region containing random irregularities which are far from the transmitter. A solution for the scattered power as a function of the position of the receiving system is sought for large distances from the scattering volume. The central interest in this problem is the derivation of scattering cross-sections.

The bistatic problem includes the monostatic problem as a special case. In the monostatic problem the transmitter and receiver are now located in the same place; such scattering is called back-scattering.

3.2 The Solution for the Scattered Power

The geometry for the bistatic problem is shown in Figure 3.2.1. The origin of the coordinate system is in

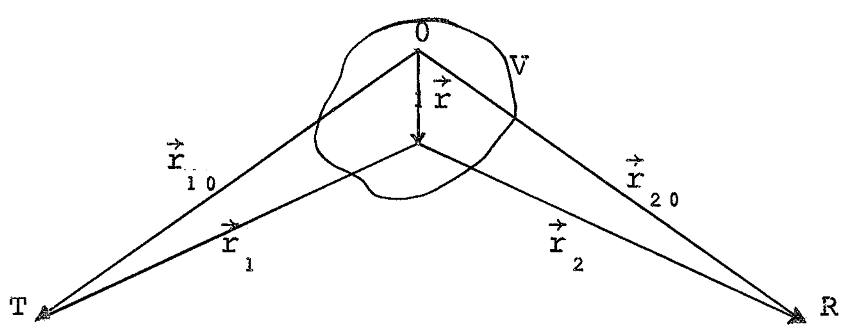


Figure 3.2.1 Geometry of the Bistatic Problem.

the scattering volume V at 0. The transmitter T is located at $\vec{r}_{10}$ and the receiver R at $\vec{r}_{20}$. The general scattering point is located at $\vec{r}$. The path from the transmitter to the scatterer is $\vec{r}_1$ and the path from the scatterer to the receiver is $\vec{r}_2$. The problem is identical to that considered by Booker and Gordon (1950) who were concerned with scattering from isotropic irregularities in an isotropic background. Booker (1956) later generalized the theory to anisotropic irregularities. We wish to generalize it further to the case of anisotropic background as well as anisotropic irregularities.

The received electric field is given by equation 2.2.6

$$\vec{E}_R(\vec{r}_{20}) = -k_0^2 \int_V \Delta X(\vec{r})\ \overline{\overline{\Gamma}}\ (\vec{r}_{20}|\vec{r}) \cdot \overline{\overline{M}} \cdot \vec{E}_i(\vec{r})\ d\vec{r} \qquad 3.2.1$$

Let the incident field be a characteristic wave

$$\vec{E}_i(\vec{r}) = A_0\ \vec{a}_i \frac{e^{j\vec{k}_i \cdot \vec{r}_1}}{r_1} \qquad 3.2.2$$

where A_0 is the amplitude of the incident wave and $\vec{k}_i$ and $\vec{a}_i$ are characteristic wave parameters associated with a ray direction $\vec{r}_1$. The Green's function $\overline{\overline{\Gamma}}$ given by equation 2.5.4 is

$$\overline{\overline{\Gamma}}(\vec{r}_{20}|r) = \frac{1}{4\pi} \sum_s \frac{d_s\ \vec{a}_s\ \vec{a}_s^{\,*}}{k_s \sec \alpha_s \sqrt{|C_s|}}\ \frac{e^{-j\vec{k}_s \cdot (\vec{r}_{20} - \vec{r})}}{|\vec{r}_{20} - \vec{r}|}$$

So equation 3.2.1 becomes

$$\vec{E}_R(\vec{r}_{20}) = -\frac{k_0^2 A_0}{4\pi} \sum_s \int_V \frac{d_s(\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i)\vec{a}_s}{k_s \sec\alpha_s \sqrt{|C_s|}} \frac{\exp -j[\vec{k}_s \cdot (\vec{r}_{20}-\vec{r}) - \vec{k}_i \cdot (\vec{r}_{10}-\vec{r})]}{|\vec{r}_{10}-\vec{r}|\ |\vec{r}_{20}-\vec{r}|} \cdot \Delta X(\vec{r}) d\vec{r} \qquad 3.2.3$$

where $\vec{r}_1 = \vec{r}_{10}-\vec{r}$ has been used. The integrand of equation 3.2.3 is an extremely complicated function of $\vec{r}$. Therefore, the solution given by equation 3.2.3 is only formal, since explicit evaluation in this general case does not seem possible. Thus some approximations are necessary to solve this equation.

The incident wave parameters are functions of the ray direction $\vec{r}_1 = \vec{r}_{10}-\vec{r}$ and the scattered wave parameters are functions of the ray direction $\vec{r}_2 = \vec{r}_{20}-\vec{r}$, where $\vec{r}$ varies over the scattering volume V. If r_{10} and r_{20} are very large compared to the volume V, then as $\vec{r}$ varies over the volume V the changes in the ray directions $\vec{r}_1$ and $\vec{r}_2$ will be very small. For most media considered here these small changes in ray direction will not have much effect on the amplitude portion of equation 3.2.3. Therefore assume that d_s, $\vec{a}_s$, $\vec{a}_i$, k_s, α_s, C_s, $|\vec{r}_{10}-\vec{r}|$ and $|\vec{r}_{20}-\vec{r}|$ are independent of $\vec{r}$. More will be said about these approximations in section 3.6.

Therefore equation 3.2.3 becomes

$$\vec{E}_R(\vec{r}_{20}) \approx -\frac{k_0^2 A_0}{4\pi} \sum_s \frac{d_s(\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i)\vec{a}_s}{k_s \sec\alpha_s \sqrt{|C_s|}\, r_{10}\, r_{20}} \int_V \Delta X(\vec{r}) \cdot \exp -j[\vec{k}_s(\vec{r}_{20}-\vec{r}) - \vec{k}_i \cdot (\vec{r}_{10}-\vec{r})] d\vec{r} \qquad 3.2.4$$

Let

$$A_s = -\frac{k_0^2 A_0}{4\pi} \frac{d_s(\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i)}{k_s \sec \alpha_s \sqrt{|C_s|} \, r_{10} r_{20}} \qquad 3.2.5$$

Thus equation 3.2.4 can be written as

$$\vec{E}_R(\vec{r}_{20}) = \sum_s A_s \vec{a}_s \int_V \Delta X(\vec{r}) \exp -j[\vec{k}_s \cdot (\vec{r}_{20} - \vec{r}) - \vec{k}_i \cdot (\vec{r}_{10} - \vec{r})] d\vec{r} \qquad 3.2.6$$

Equation 3.2.6 is still a complicated function of $\vec{r}$ since $\hat{k}_s$ and $\hat{k}_i$ depend upon $\vec{r}$. Assume that $\hat{k}_s$ and $\hat{k}_i$ are independent of $\vec{r}$, using the same reasoning as before but being more restrictive as to the type of medium and/or the size of the volume V. There are requirements on the minimum size of V (see discussion in the paragraph about $\langle \Delta X(\vec{r}_1) \Delta X(\vec{r}_2) \rangle$ in connection with equation 3.2.10) which will not allow the volume to be made arbitrarily small. Thus equation 3.2.6 becomes

$$\vec{E}_R(\vec{r}_{20}) = \sum_s A_s \vec{a}_s \exp[-j(\vec{k}_s \cdot \vec{r}_{20} - \vec{k}_i \cdot \vec{r}_{10})] \int_V \Delta X(\vec{r}) \cdot \exp[j(\vec{k}_s - \vec{k}_i) \cdot \vec{r}] d\vec{r} \qquad 3.2.7$$

The magnetic field corresponding to $\vec{E}_R$ can be found by using one of Maxwell's equations $\nabla \times \vec{E} = -j\omega\mu\vec{H}$, where the medium has been assumed to be isotropic in its magnetic properties. The magnetic field is

$$\vec{H}_R(\vec{r}_{20}) = j \frac{1}{\omega\mu} \nabla \times \vec{E}_R(\vec{r}_{20})$$

where ∇ operates on $\vec{r}_{20}$ coordinates. To $0(\frac{1}{r_{20}^2})$, $\vec{H}_R$ is

$$\vec{H}_R(\vec{r}_{20}) = \sum_s \frac{A_s}{\omega\mu} (\vec{k}_s \times \vec{a}_s)$$

$$\cdot \exp[-j(\vec{k}_s \cdot \vec{r}_{20} - k_i \cdot \vec{r}_{10})] \int_V \Delta X(\vec{r}) \exp[j(\vec{k}_s - \vec{k}_i) \cdot \vec{r}] d\vec{r} \qquad 3.2.8$$

The received Poynting vector is

$$\vec{P}_R(\vec{r}_{20}) = \frac{1}{2} \mathrm{Re}[\vec{E}_R(\vec{r}_{20}) \times \vec{H}_R^* (\vec{r}_{20})]$$

$$\vec{P}_R(\vec{r}_{20}) = \frac{1}{2\omega\mu} \mathrm{Re}\{ \Sigma_{s,t} A_s A_t^* [\vec{a}_s \times (\vec{k}_t \times \vec{a}_t^*)] e^{-j(\vec{k}_s - \vec{k}_t)\cdot\vec{r}_{20}}$$

$$\iint_V \Delta X(\vec{r}_1) \Delta X(\vec{r}_2) \exp j[(\vec{k}_s \cdot \vec{r}_1 - \vec{k}_t \cdot \vec{r}_2) - \vec{k}_i \cdot (\vec{r}_1 - \vec{r}_2)] d\vec{r}_1 d\vec{r}_2 \}$$

Take the average value of this equation

$$\langle \vec{P}_R(\vec{r}_{20}) \rangle = \frac{1}{2\omega\mu} \mathrm{Re} \{ \sum_{s,t} A_s A_t^* [\vec{a}_s \times (\vec{k}_t \times \vec{a}_t^*)] e^{-j(\vec{k}_s - \vec{k}_t)\cdot\vec{r}_{20}} I \} \qquad 3.2.9$$

where

$$I = \iint_V \langle \Delta X(\vec{r}_1) \Delta X(\vec{r}_2) \rangle \exp\{ j[(\vec{k}_s \cdot \vec{r}_1 - \vec{k}_t \cdot \vec{r}_2) - \vec{k}_i \cdot (\vec{r}_1 - \vec{r}_2)] \} d\vec{r}_1 d\vec{r}_2 \qquad 3.2.10$$

For this problem the average value $\langle \Delta X(\vec{r}_1) \Delta X(\vec{r}_2) \rangle$ is chosen to depend only on the relative coordinates $\vec{r}_1 - \vec{r}_2$. This follows if the random variable $\Delta X(\vec{r})$ is assumed to be homogeneous. This assumption can be strictly true only if the medium is infinite, which is not the case in this problem. If the volume V is very large in terms of the largest correlation length associated with the problem, then the edge effects due to the non-homogeneity at the boundaries will be very small and the random variable can be assumed to be homogeneous. Let the average value be given by

$$<\Delta X(\vec{r}_1)\Delta X(\vec{r}_2)> = <(\Delta X)^2> B(\vec{r}_1-\vec{r}_2)$$

where B is the normalized correlation function of ΔX. With this assumption equation 3.2.10 can be written as

$$I = <(\Delta X)^2> \iint_V B(\vec{r}_1-\vec{r}_2)\exp\{j[(\vec{k}_s\cdot\vec{r}_1-\vec{k}_t\cdot\vec{r}_2)-\vec{k}_i\cdot(\vec{r}_1-\vec{r}_2)]\}d\vec{r}_1 d\vec{r}_2 \qquad 3.2.11$$

Now expand the wave vector $\vec{k}_t$ as a sum of $\vec{k}_s$ and a remainder vector $\vec{b}_t$. Then equation 3.2.11 can be written as

$$I = <(\Delta X)^2> \iint_V B(\vec{r}_1-\vec{r}_2)\exp[j(\vec{k}_s-\vec{k}_i)\cdot(\vec{r}_1-\vec{r}_2)]\exp[-j\,\vec{b}_t\cdot\vec{r}_2]d\vec{r}_1 d\vec{r}_2 \qquad 3.2.12$$

This equation can be simplified by making a change of variables. Let $\vec{\eta} = \vec{r}_1 - \vec{r}_2$ and $\zeta = \vec{r}_2$. The Jacobian of the transformation is

$$J = \frac{\partial(\vec{r}_1,\vec{r}_2)}{\partial(\vec{\eta},\vec{\zeta})} = \begin{bmatrix} \frac{\partial\vec{r}_1}{\partial\vec{\eta}} & \frac{\partial\vec{r}_1}{\partial\vec{\zeta}} \\ \frac{\partial\vec{r}_2}{\partial\vec{\eta}} & \frac{\partial\vec{r}_2}{\partial\vec{\zeta}} \end{bmatrix} = 1$$

Then equation 3.2.12 can be written as

$$I = <(\Delta X)^2> \iint_V B(\vec{\eta})\ \exp[j(\vec{k}_s-\vec{k}_i)\cdot\vec{\eta}-j\,\vec{b}_t\cdot\vec{\zeta}]d\vec{\eta}\ d\vec{\zeta} \qquad 3.2.13$$

Now consider the function $F(\vec{b}_t) = \int_V \exp[-j\,\vec{b}_t\cdot\vec{\zeta}]d\vec{\zeta}$. $F(\vec{b}_t)$ is a function which has a sharp peak at $\vec{b}_t = 0$ and oscillates and decreases rapidly away from $\vec{b}_t = 0$ if $\frac{1}{b_t} << \sqrt[3]{V}$ (Tatarski, 1961). Since $\vec{b}_t$ is the difference between wave vectors of two different modes or saddle points it will be of the order $1/\lambda$. If the dimensions of the volume V are large in terms of wavelengths, then $F(\vec{b}_t)$ will be approximately equal to zero unless $\vec{b}_t \approx 0$. But $\vec{b}_t$ can be zero only if $t = s$. Thus $F(0) = V$ and $t = s$.

Therefore equation 3.2.13 becomes

$$I = V\langle(\Delta X)^2\rangle \int_V B(\vec{\eta}) \exp[j(\vec{k}_s-\vec{k}_i)\cdot\vec{\eta}]d\vec{\eta} \qquad 3.2.14$$

Since V is very large in terms of correlation lengths, the integral over the volume can be replaced by an integral over all space without appreciably changing the value of I. Therefore equation 3.2.14 becomes

$$I = V\langle(\Delta X)^2\rangle \int_{-\infty}^{\infty} B(\vec{\eta}) \exp[j(\vec{k}_s-\vec{k}_i)\cdot\vec{\eta}]d\vec{\eta} \qquad 3.2.15$$

The integral in the above equation is just the spectrum Φ of the correlation function B. So finally

$$I = V\langle(\Delta X)^2\rangle \, \Phi_B \, (\vec{k}_s-\vec{k}_i) \qquad 3.2.16$$

Thus the average Poynting vector at the receiver given by equation 3.2.9 becomes

$$\langle\vec{P}_R(\vec{r}_{20})\rangle = \frac{V\langle(\Delta X)^2\rangle}{2\omega\mu} \sum_s |A_s|^2 \, \Phi_B(\vec{k}_s-\vec{k}_i)\,\mathrm{Re}[\vec{a}_s \times (\vec{k}_s \times \vec{a}_s^*)] \qquad 3.2.17$$

Consider the factor

$$\mathrm{Re}[\vec{a}_s \times (\vec{k}_s \times \vec{a}_s^*)] = \mathrm{Re}[|\vec{a}_s|^2\vec{k}_s - (\vec{k}_s\cdot\vec{a}_s)\vec{a}_s^*] \qquad 3.2.18$$

In the coordinate system shown in Figure 2.6.1

$$\vec{a} = \frac{1}{\sqrt{1+|R|^2}} [R,\ 1,\ RQ]$$

where R and Q are both imaginary in the propagating region. Compute

$$|\vec{a}|^2\ \vec{k} = \frac{\hat{z}k}{1+|R|^2} (|R|^2 + 1 + |R|^2|Q|^2)$$

which is real, and

$$(\vec{k}\cdot\vec{a})\vec{a}^* = \frac{kRQ}{1+|R|^2} [R^*,\ 1,\ R^*Q^*]$$

Thus

$$\mathrm{Re}[\vec{a} \times (\vec{k}\times\vec{a}^*)] = k\ [0,\ -\frac{RQ}{1+|R|^2},\ 1] \qquad 3.2.19$$

Figure 3.2.2 shows the relationship between the average

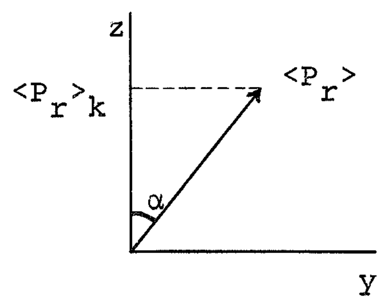


Figure 3.2.2 Power in the directions of the ray and group velocity.

Poynting vector and the component in the direction of the wave normal. From Figure 3.2.2

$$\langle\vec{P}_R\rangle = \langle P_R\rangle_k \sec\alpha\ \hat{r}_{20}$$

Thus from equations 3.2.17 - 3.2.19 and the above, equation 3.2.17 simplifies to

$$\langle \vec{P}_R(\vec{r}_{20})\rangle = \frac{V\langle(\Delta X)^2\rangle}{2\omega\mu} \sum_s |A_s|^2\, k_s \sec\alpha_s\, \Phi_B\,(\vec{k}_s-\vec{k}_i)\,\hat{r}_{20} \qquad 3.2.20$$

Now find the factor $|A_s|^2$ from equation 3.2.5

$$|A_s|^2 = \frac{k_0^4 A_0^2}{(4\pi)^2}\, \frac{|d_s|^2 |\vec{a}_s^{\,*}\cdot\overline{\overline{M}}\cdot\vec{a}_i|^2}{k_s^2 \sec^2\alpha_s\, |C_s|\, r_{10}^2 r_{20}^2}$$

but $|d_s|^2 = 1$ for all possible surfaces.

The average power flow at the receiver is just the Poynting vector in the direction $\hat{r}_{20}$. Its magnitude now simplifies to

$$\langle P_R(\vec{r}_{20})\rangle = \frac{k_0^4 A_0^2 \langle(\Delta X)^2\rangle V}{32\pi^2\omega\mu r_{10}^2 r_{20}^2} \sum_s \frac{|\vec{a}_s^{\,*}\cdot\overline{\overline{M}}\cdot\vec{a}_i|^2}{k_s \sec\alpha_s\, |C_s|}\, \Phi_B(\vec{k}_s-\vec{k}_i) \qquad 3.2.21$$

This is the expression of interest. The expression for the scattering cross section is derived in the next section.

3.3 The Scattering Cross Section σ_s^i

The scattering cross section σ_s^i is defined as the power scattered per unit solid angle from a unit volume for the mode and saddle point s, with unit incident power in mode and saddle point i (Cohen, 1962)

The incident electric field is given by equation 3.2.2 as

$$\vec{E}_i(\vec{r}) = A_0\vec{a}_i(\vec{r}_1)\, \frac{\exp[j\,\vec{k}_i(\vec{r}_1)\cdot(\vec{r}_{10}-\vec{r})]}{|\vec{r}_{10}-\vec{r}|}$$

Using the assumptions of the preceding section this equation becomes

$$\vec{E}_i(\vec{r}) = A_0\vec{a}_i(\vec{r}_{10})\, \frac{\exp[j\,\vec{k}_i(\vec{r}_{10})\cdot(\vec{r}_{10}-\vec{r})]}{r_{10}} \qquad 3.3.1$$

The corresponding magnetic field can be obtained by using the same method as in the preceding section,

$$\vec{H}_i(\vec{r}) = \frac{A_0}{\omega\mu}\vec{k}_i \times \vec{a}_i \frac{\exp[j\,\vec{k}_i\cdot(\vec{r}_{i0}-\vec{r})]}{r_{10}}$$

Thus, the incident Poynting vector is

$$\vec{P}_i = \frac{A_0^2}{2\omega\mu r_{10}^2}\,\mathrm{Re}[\vec{a}_i \times (k_i \times \vec{a}_i^{\,*})]$$

Thus using the same procedure as in the preceding section the incident power is

$$P_i = \frac{A_0^2}{2\omega\mu r_{10}^2}\,k_i \sec\alpha_i \qquad 3.3.2$$

The scattering cross section is

$$\sigma_s^i = \frac{\langle P_R(\vec{r}_{20})\rangle_s\, r_{20}^2}{V\,P_i}$$

Using equations 3.2.21 and 3.3.2 this equation becomes

$$\sigma_s^i = \frac{k_0^4}{16\pi^2}\,\frac{|\vec{a}_s^{\,*}\cdot\overline{\overline{M}}\cdot\vec{a}_i|^2}{k_i k_s|C_s|}\cos\alpha_i\cos\alpha_s\langle(\Delta X)^2\rangle\Phi_B(\vec{k}_s-\vec{k}_i) \qquad 3.3.3$$

Therefore the scattering cross section from mode i into mode s is obtained by summing over the saddle points of s to get

$$\sigma_s^i = \frac{k_0^4}{16\pi^2}\sum\frac{|\vec{a}_s^{\,*}\cdot\overline{\overline{M}}\cdot\vec{a}_i|^2}{k_i k_s|C_s|}\cos\alpha_i\cos\alpha_s\,\langle\Delta X^2\rangle\Phi_B(\vec{k}_s-\vec{k}_i) \qquad 3.3.4$$

Equation 3.3.3 shows that the scattering cross section divides nicely into a portion which depends only on the geometry of the problem and the properties of the medium, and a portion which contains the properties of the random process ΔX. Thus equation 3.3.3 can be written as

$$\sigma_s^i = \frac{k_0^4}{16\pi^2} G_s^i (\vec{r}_{10},\vec{r}_{20},X,Y) \; S_s^i (\vec{r}_{10},\vec{r}_{20},X,Y) \qquad 3.3.5$$

where

$$G_s^i = \frac{|\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i|^2}{k_i k_s |C_s|} \cos\alpha_i \cos\alpha_s \qquad 3.3.6$$

$$S_s^i = \langle(\Delta X)^2\rangle \; \Phi_B(\vec{k}_s - \vec{k}_i) \qquad 3.3.7$$

The properties of these two functions will be discussed separately in the next two sections.

3.4 The Geometric Factor G_s^i

The only information necessary to calculate G_s^i is given by the geometry and the parameters of the background medium. The orientation of the transmitted ray path and the scattered ray path to the magnetic field and the type of mode determine k_i, k_s, C_s, $\vec{a}_i$, $\vec{a}_s$, α_i and α_s. The tensor $\overline{\overline{M}}$ depends upon the orientation and strength of the D. C. magnetic field.

If equation 2.7.6 is used for $|C_s|$, equation 3.3.6 can be written as

$$G_s^i = |\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i|^2 \frac{\sin\theta_s}{\sin\rho_s \cos\alpha_s} \left|\frac{d\theta_s}{d\rho_s}\right| \frac{k_s^2}{k_i \sec\alpha_i \; k_s \sec\alpha_s} \qquad 3.4.1$$

This equation can be divided into three factors which will now be discussed.

The factor $\frac{k_s^2}{k_i \sec\alpha_i \; k_s \sec\alpha_s}$ arises from the ratio of k_s^2 to the product of the intrinsic powers ($\mathrm{Re}(\vec{a} \times \vec{k} \times \vec{a}^*)$)

of the incident and scattered modes. This can be seen from equations 3.2.18 and 3.2.19 and Figure 3.2.2. As the medium approaches an isotropic medium $(Y \to 0)$ α_s and α_i approach zero and k_s approaches k_i. Therefore this factor reduces to unity for an isotropic medium.

The factor $\frac{\sin\theta_s}{\sin\rho_s \cos\alpha_s} \left|\frac{d\theta_s}{d\rho_s}\right|$ is related to the properties of the refractive index surface at the saddle point s defined by ρ_s. The term $\left|\frac{d\theta_s}{d\rho_s}\right|$ is a function of the angle α_s and the second derivative of the surface, and thus is related to the curvature of the surface at the saddle point. As the medium tends to isotropy, θ_s tends to ρ_s so α_s approaches zero and the surface tends to a circle for which $\frac{d\theta}{d\rho}$ is unity. Thus this factor also reduces to unity for an isotropic medium.

The final factor $|\vec{a}_s^{\,*} \cdot \overline{\overline{M}} \cdot \vec{a}_i|^2$ has the following physical meaning. The electric polarization induced by the incident field $\vec{a}_i$ due to presence of electron density irregularities is $\Delta X \overline{\overline{M}} \cdot \vec{a}_i$. Therefore, $|\vec{a}_s^{\,*} \cdot \overline{\overline{M}} \cdot \vec{a}_i|$ is proportional to the projection of the induced polarization on the scattered field corresponding to mode s. As the medium becomes isotropic by the decrease of Y, $\overline{\overline{M}}$ becomes the unity tensor $\overline{\overline{I}}$, and $|\vec{a}_s^{\,*} \cdot \overline{\overline{M}} \cdot \vec{a}_i|^2$ becomes $|\vec{a}_s^{\,*} \cdot \vec{a}_i|^2$.

As Y approaches zero, R, as given by equation 2.6.4b, becomes $\mp j$ and Q, given by equation 2.6.4c, becomes zero. For this special case the characteristic vector $\vec{a}$ is

$$\vec{a} = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm j \sin \gamma - \cos \rho \cos \gamma \\ \pm j \cos \gamma - \cos \rho \sin \gamma \\ \sin \rho \end{bmatrix} \qquad 3.4.2$$

where the fact that $\theta = \rho$ for the isotropic case has been used in equation 4 of appendix A1. The angles ρ and γ are the spherical angles of the ray direction of the wave.

The geometry of this problem in terms of these angles is shown in Figure 3.4.1. Thus the product $\vec{a}_s^* \cdot \vec{a}_i$ is

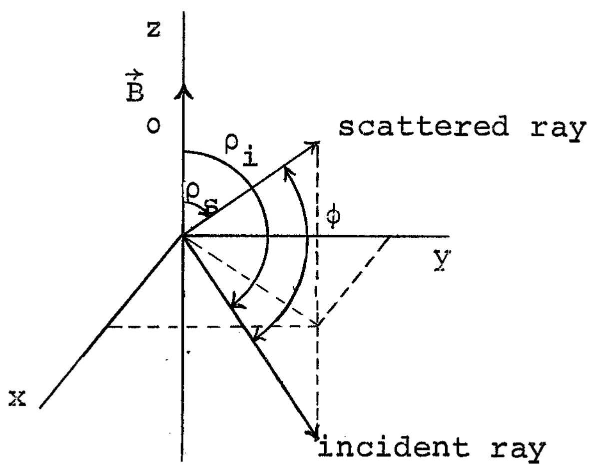


Figure 3.4.1 The relation between the incident and scattered rays.

$$\vec{a}_s^* \cdot \vec{a}_i = \frac{1}{2} \begin{bmatrix} (-1)^{\ell} j \sin \gamma - \cos \rho_s \cos \gamma \\ (-1)^{\ell+1} j \cos \gamma - \cos \rho_s \sin\gamma \\ \sin \rho_s \end{bmatrix} \cdot \begin{bmatrix} (-1)^{h+1} j \sin\gamma - \cos\rho_i \cos\gamma \\ (-1)^{h} j \cos\gamma - \cos\rho_i \sin \gamma \\ \sin \rho_i \end{bmatrix}$$

where h = 1 for the ordinary mode and h = 2 for the extra-ordinary mode. The same is true of ℓ.

$$\vec{a}_s^* \cdot \vec{a}_i = \frac{1}{2} \; [(-1)^{h+\ell} + \cos \rho_i \cos \rho_s + \sin \rho_i \sin \rho_s]$$

$$= \frac{1}{2} \; [(-1)^{h+\ell} + \cos \phi] \qquad 3.4.3$$

where $\phi = \rho_i - \rho_s$ is the scattering angle.

If the incident field is a superposition of characteristic waves, $\vec{a}_s^* \cdot \vec{a}_i$ becomes $\sum_i A_i (\vec{a}_s^* \cdot \vec{a}_i)$. If the incident wave is assumed to be a linearly polarized wave whose electric vector makes an angle η with the plane defined by the incident and scattered rays then the amplitudes A_i are

$$A_1 = \cos\eta + j\sin\eta$$

$$A_2 = \cos\eta - j\sin\eta$$

Thus

$$\sum_i A_i (\vec{a}_s^* \cdot \vec{a}_i)$$

$$= \frac{1}{2}[(\cos\eta + j\ \sin\eta)(-(-1)^{\ell} + \cos\phi) + (\cos\eta - j\ \sin\eta)((-1)^{\ell} + \cos\phi)]$$

$$= \cos\phi\cos\eta - j(-1)^{\ell}\sin\eta$$

Therefore for the isotropic case

$$|\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i|^2 = \sin^2\eta + \cos^2\phi\cos^2\eta \qquad 3.4.4$$

which is exactly equal to the $\sin^2\chi$ of Booker and Gordon (1950) for the geometry used here.

The above analysis and equation 3.3.5 show that the scattering cross-section for the isotropic medium is

$$\sigma = \frac{k_0^4 \langle(\Delta X)^2\rangle}{16\pi^2}(\sin^2\eta + \cos^2\phi\cos^2\eta)\ \Phi[k(\hat{r}_s - \hat{r}_i)] \qquad 3.4.5$$

This is exactly equal to equation 17 of Booker (1956) when written for the geometry used here.

Two special cases for the geometric factor are of special interest. The first is the forward scattering case in which the scattering angle ϕ (see Figure 3.4.1) is equal to zero. This case occurs when one is looking at the transmitter through the scattering medium. The other case is the back-scatter case for which the scattering angle ϕ is equal to π. This is the radar case which is always of much practical interest. The geometric factor for these two cases is identical because of the symmetry of the refractive index surfaces considered here. These surfaces are surfaces of revolution about the magnetic field which results in the symmetric nature of these surfaces. For convenience, these two cases will be referred to collectively as the forward-scatter case.

Before discussing the forward-scatter case in general two special cases will be discussed. For the case where the scattering angle ϕ is zero the only angle necessary to describe the geometry is the angle ρ that the ray makes with the magnetic field (see Figure 3.4.1). The first case is longitudinal propagation for which ρ is zero. The second case is transverse propagation for which ρ is $\pi/2$. For these two cases the relationship among the vectors is simple enough so that reasonably simple equations result.

First find $\frac{d\rho}{d\theta}$, to be used in equation 3.4.1. From 2.7.5

$$\frac{d\rho}{d\theta} = 1 + \sin^2 \alpha - \frac{n''}{n} \cos^2 \alpha \qquad 3.4.6$$

Thus in order to find $\frac{d\rho}{d\theta}$ one must know α and $\frac{n''}{n}$. From equation 2.6.3

$$n^2 = 1 - X/(1-T) \qquad 3.4.7$$

where
$$T = \frac{Y^2 \sin^2 \theta}{2(1-X)} \mp \left[\frac{Y^4 \sin^4 \theta}{4(1-X)^2} + Y^2 \cos^2 \theta\right]^{1/2} \qquad 3.4.8$$

Therefore

$$n' = - \frac{XT'}{2n(1-T)^2} \qquad 3.4.9$$

$$T' = \frac{2Y^2 \sin\theta\cos\theta}{2(1-X)} \mp \frac{1}{2}\left[\frac{Y^4 \sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1/2} \left[\frac{4Y^4 \sin^3\theta\cos\theta}{4(1-X)^2} - 2Y^2\sin\theta\cos\theta\right]$$

$$= \frac{Y^2 \sin\theta\cos\theta}{1-X} \left\{1 \mp \left[\frac{Y^4 \sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1/2} \left[\frac{Y^2\sin^2\theta}{2(1-X)} - (1-X)\right]\right\} \qquad 3.4.10$$

So if θ equals 0 or $\pi/2$, T' is zero and therefore n' is zero. From equation 2.7.3 $\tan \alpha = - \frac{n'}{n}$ and thus, since n' is zero at $\theta = 0$ or $\pi/2$, α is zero at $\theta = 0$ or $\pi/2$. Therefore for θ equal to 0 or $\pi/2$

$$\frac{d\rho}{d\theta} = 1 - \frac{n''}{n} \qquad 3.4.11$$

Now find $\frac{n''}{n}$. From equation 3.4.9

$$n'' = -\frac{XT''}{2n(1-T)^2} - \frac{XT'^2}{n(1-T)^3} - \frac{X^2T'^2}{4n^3(1-T)^4} \qquad 3.4.12$$

but T' is zero so

$$n'' = -\frac{XT''}{2n(1-T)^2}$$

$$\frac{n''}{n} = -\frac{XT''}{2n^2(1-T)^2}$$

but from equation 3.4.7 $n^2(1-T) = 1 - X - T$, so

$$\frac{n''}{n} = -\frac{XT''}{2(1-X-T)(1-T)} \qquad 3.4.13$$

From equation 3.4.10

$$T'' = \frac{Y^2(\cos^2\theta-\sin^2\theta)}{1-X}\left\{1 \mp \left[\frac{Y^4\sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1/2} \left[\frac{Y^2\sin^2\theta}{2(1-X)} - (1-X)\right]\right\}$$

$$+ \frac{Y^2\sin\theta\cos\theta}{1-X}\left\{\pm\frac{1}{2}\left[\frac{Y^4\sin^4\theta}{4(1-X)^2}+Y^2\cos^2\theta\right]^{-3/2} \left[\frac{Y^2\sin^2\theta}{2(1-X)} - (1-X)\right]\right.$$

$$\cdot \left[\frac{4Y^4\sin^3\theta\cos\theta}{4(1-X)^2} - 2Y^2\sin\theta\cos\theta\right]$$

$$\left.\mp \left[\frac{Y^4\sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1/2} \frac{2Y^2\sin\theta\cos\theta}{2(1-X)}\right\}$$

$$T'' = \frac{Y^2(\cos^2\theta-\sin^2\theta)}{(1-X)} \left\{ 1 \mp \left[\frac{Y^4\sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1/2} \left[\frac{Y^2\sin^2\theta}{2(1-X)} - (1-X)\right]\right\}$$

$$\mp \frac{Y^4\sin^2\theta\cos^2\theta}{(1-X)^2} \left[\frac{Y^4\sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1/2}$$

$$\cdot \left\{1 - \left[\frac{Y^4\sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1} \left[\frac{Y^2\sin^2\theta}{2(1-X)} - (1-X)\right]^2\right\} \quad 3.4.14$$

For θ equal to 0 or $\pi/2$ the second term is zero so that

$$T'' = \frac{Y^2(\cos^2\theta-\sin^2\theta)}{1-X} \left\{ 1 \mp \left[\frac{Y^4\sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1/2} \left[\frac{Y^2\sin^2\theta}{2(1-X)} - (1-X)\right]\right\}$$

Thus from equations 3.4.11 and 3.4.13

$$\frac{d\rho}{d\theta} = 1+\frac{XY^2(\cos^2\theta-\sin^2\theta)}{2(1-X)(1-T)(1-X-T)} \left\{ 1 \mp \left[\frac{Y^4\sin^4\theta}{4(1-X)^2} + Y^2\cos^2\theta\right]^{-1/2} \left[\frac{Y^2\sin^2\theta}{2(1-X)} - (1-X)\right]\right\} \quad 3.4.15$$

From this, one can compute the value of

$$\frac{\sin\rho_s\cos\alpha_s}{\sin\theta_s}\,\frac{d\rho}{d\theta}\,\frac{k_i\sec\alpha_i\,k_s\sec\alpha_s}{k_s^2} \quad 3.4.16$$

for $\theta = 0$ and $\pi/2$. For $\theta = 0$ and $\pi/2$, $\alpha_s = \alpha_i = 0$ since n' is zero. For $\theta = 0$

$$\lim_{\theta\to 0} \frac{\sin\rho_s}{\sin\theta_s} = \frac{d\rho}{d\theta}$$

Thus from the above and equations 3.4.8 and 3.4.15 equation 3.4.16 becomes

$$\frac{\sin \rho_s \cos \alpha_s}{\sin \theta_s} \frac{d\rho}{d\theta} \frac{k_i \sec\alpha_i \; k_s \sec\alpha_s}{k_s^2} = \frac{k_i}{k_s}\left[1 \pm \frac{XY}{2(1-X)(1\pm Y)}\right]^2 \qquad 3.4.17$$

For $\theta = \pi/2$ equation 3.4.16 becomes

$$\frac{\sin \rho_s \cos \alpha_s}{\sin \theta_s} \frac{d\rho}{d\theta} \frac{k_i \sec\alpha_i \; k_s \sec\alpha_s}{k_s^2} = \frac{k_i}{k_s}\left[1 \mp \frac{X(1-X)}{1-X-(Y^2/2)(1\mp 1)}\right] \qquad 3.4.18$$

Now $|\vec{a}_s^{*}\cdot\overline{\overline{M}}\cdot\vec{a}_i|^2$ must be computed. In the coordinate system of Figure 3.4.1, the quantity M is

$$\overline{\overline{M}} = \begin{bmatrix} \dfrac{1}{1-Y^2} & \dfrac{jY}{1-Y^2} & 0 \\ -\dfrac{jY}{1-Y^2} & \dfrac{1}{1-Y^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 3.4.19$$

From equation 2.6.4 for $\theta = 0$, $R = \mp j$ and $Q = 0$. The characteristic field for $\theta = 0$ can be computed from equation A1.4 as

$$\vec{a} = \frac{1}{\sqrt{2}}\,[-1, \mp j, 0] \qquad \theta = 0 \qquad 3.4.20$$

where γ has been set equal to zero without loss of generality. The characteristic field for transverse propagation ($\theta = \pi/2$) cannot be obtained in the same way because $R = 0$ or ∞. The expressions for these fields are given by Papas in section 6.6 with some modifications as

$$\vec{a}_1 = [0, 0, 1] \qquad 3.4.21a$$

$$\vec{a}_2 = [\frac{XY}{1-X-Y^2}, -j, 0] \qquad 3.4.21b$$

where propagation along the x axis has been assumed.

For $\theta = 0$ the product $\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i$ is

$$\frac{1}{2}[1, (-1)^{\ell} j, 0] \begin{bmatrix} \frac{1}{1-Y^2} & \frac{jY}{1-Y^2} & 0 \\ -\frac{jY}{1-Y^2} & \frac{1}{1-Y^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ (-1)^{h+1} j \\ 0 \end{bmatrix} = \frac{\delta_{h\ell}}{1 \pm Y} \qquad 3.4.22$$

While for $\theta = \pi/2$ the product is

$$\vec{a}_1^* \cdot \overline{\overline{M}} \cdot \vec{a}_1 = 1 \qquad 3.4.23a$$

$$\vec{a}_1^* \cdot \overline{\overline{M}} \cdot \vec{a}_2 = 0 \qquad 3.4.23b$$

$$\vec{a}_2^* \cdot \overline{\overline{M}} \cdot \vec{a}_1 = 0 \qquad 3.4.23c$$

$$\vec{a}_2^* \cdot \overline{\overline{M}} \cdot \vec{a}_2 = \frac{1}{1-Y^2}\left[1 + \frac{2XY^2}{1-X-Y^2} + \left(\frac{XY}{1-X-Y^2}\right)^2\right] \qquad 3.4.23d$$

The geometric factor G_s^i can now be computed for the special cases of longitudinal and transverse propagation for the forward-scatter geometry. Equations 3.4.1, 17, 18, 22 and 23 can now be combined to obtain G_s^i. The results are shown in Table 3.4.1, where 1 refers to the ordinary mode and 2 refers to the extraordinary mode. There is no cross mode scattering because of the complimentary nature of the characteristic waves at these two extremes. The expressions for $\theta = 0$ are symmetrical because the characteristic waves are a conjugate pair.

Table 3.4.1 The Geometric Factor for $\theta = 0$ and $\pi/2$

Geometric Factor G_s^i

i	s	$\theta = 0$	$\theta = \pi/2$
1	1	$[\frac{2(1-X)}{2(1-X)(1+Y)+XY}]^2$	$\frac{1}{1-X}$
1	2	0	0
2	1	0	0
2	2	$[\frac{2(1-X)}{2(1-X)(1-Y)-XY}]^2$	$\{\frac{1}{1-Y^2}[1+ \frac{2XY^2}{1-X-Y^2} + (\frac{XY}{1-X-Y^2})^2]\}^2 \cdot [1+\frac{X(1-X)}{1-X-Y^2}]^{-1}$

No such symmetry exists for $\theta = \pi/2$, and so the two expressions are markedly different.

While the expressions given in Table 3.4.1 are useful for computing the range of the geometric factor for some plasmas, they do not give any picture of what happens between them. These expressions also do not give any indication of the amount of cross mode scattering that may occur. If the geometry of Figure 3.4.1 is used, equation 2.2.2 for $\overline{\overline{M}}$ becomes

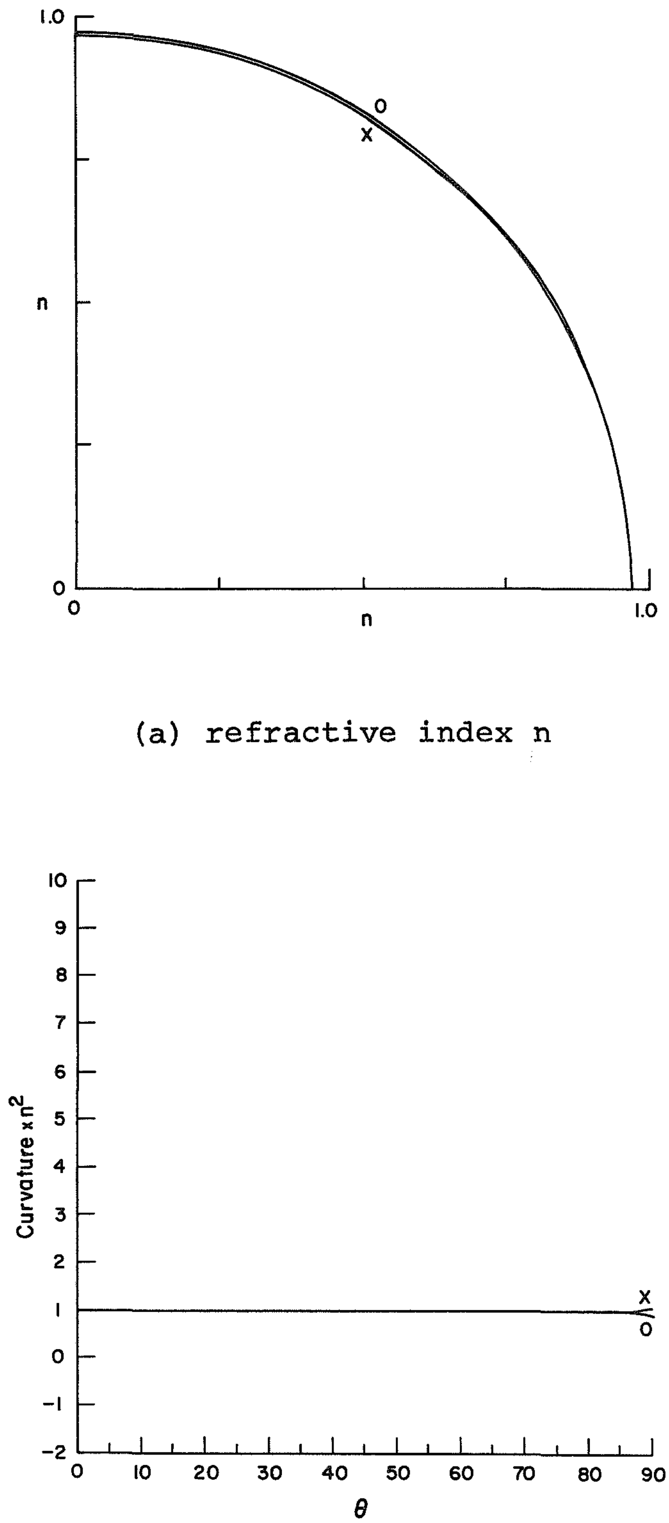
$$\overline{\overline{M}} = \frac{1}{1-Y^2} \begin{bmatrix} 1 & jY & 0 \\ -jY & 1 & 0 \\ 0 & 0 & 1-Y^2 \end{bmatrix} \qquad 3.4.24$$

The characteristic vector in this coordinate system is given by equation A1.4, with $\gamma = 0$. n is given by equation

2.6.3, C by equation 2.7.4 or 2.7.6 and α by equation 2.7.3.

The physical problem that is to be solved can be described as follows; incident energy is beamed along the z-axis and it is received on the z-axis, while the magnetic field makes an angle β with the z-axis. This is equivalent to the above geometry if the ray direction ρ is taken as β. A computer program was written to solve the equations in the coordinate system of the preceding paragraph while displaying the results in the above geometry.

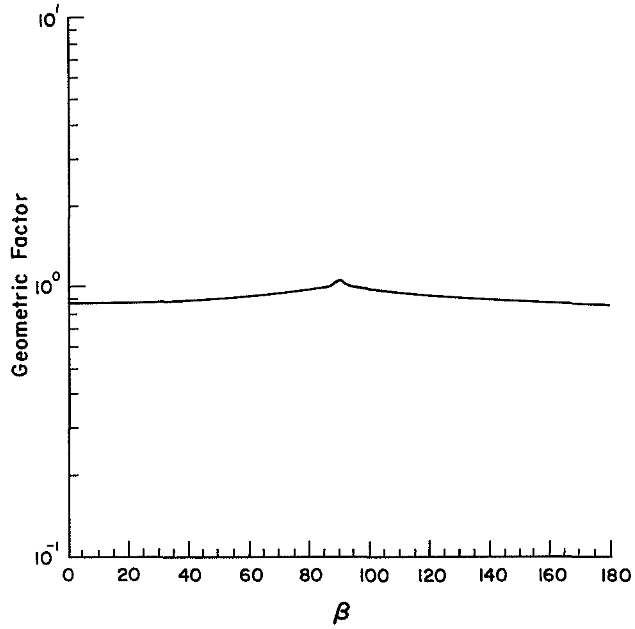
Figure 3.4.2(a) shows the refractive index as a polar plot with respect to θ ($\theta = 0$ is vertical). Figure 3.4.2(b) shows the curvature of n normalized with respect to n^2 (a circle is thus 1). Figure 3.4.3 shows the geometric factor of the forward scatter case for the plasma shown in Figure 3.4.2 as a function of β. This plasma is appropriate to the earth's ionosphere in the F-region for a frequency of 20 MHz. Section (a) shows the scattering from the ordinary mode into the ordinary mode. G starts out at the value given in Table 3.4.1 for $\theta = 0$ and slowly rises until the curvature starts to decrease (see Figure 3.4.2(b)) when it rises rapidly to the value given in the table for $\theta = \pi/2$. The exact opposite happens with the scattering from the extraordinary mode into the extra-ordinary mode shown in section d. Sections (b) and (c) show the cross mode scattering. G_s^i peaks at about 88° and



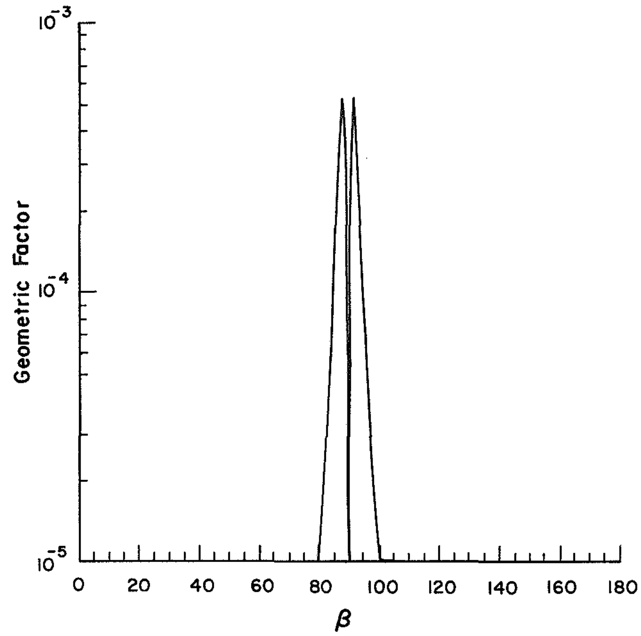
(a) refractive index n

(b) normalized curvature

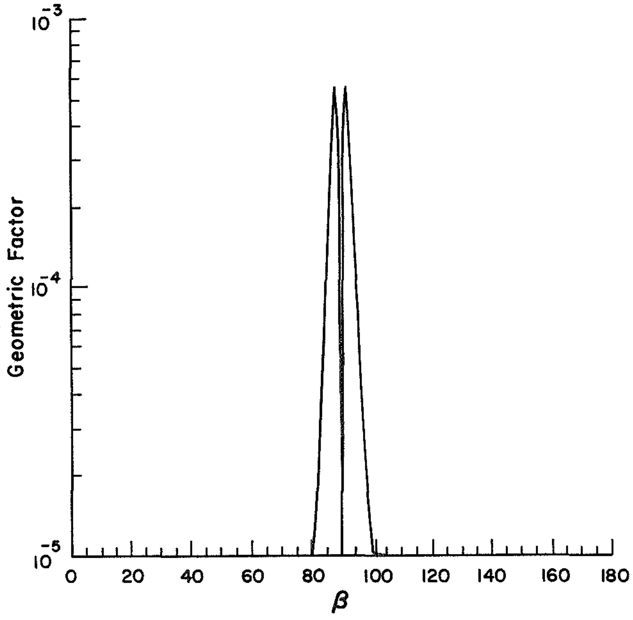
Figure 3.4.2 The refractive index and Gaussian curvature for X = 0.063 and Y = 0.075.



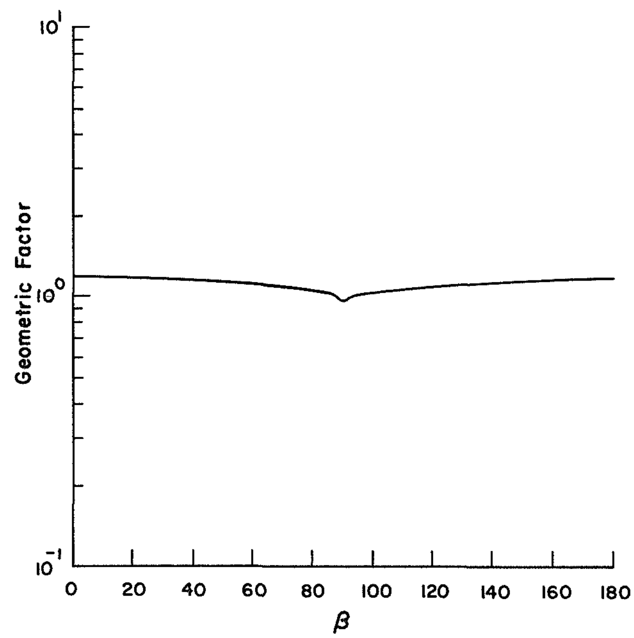
(a) ordinary to ordinary



(b) ordinary to extraordinary



(c) extraordinary to ordinary



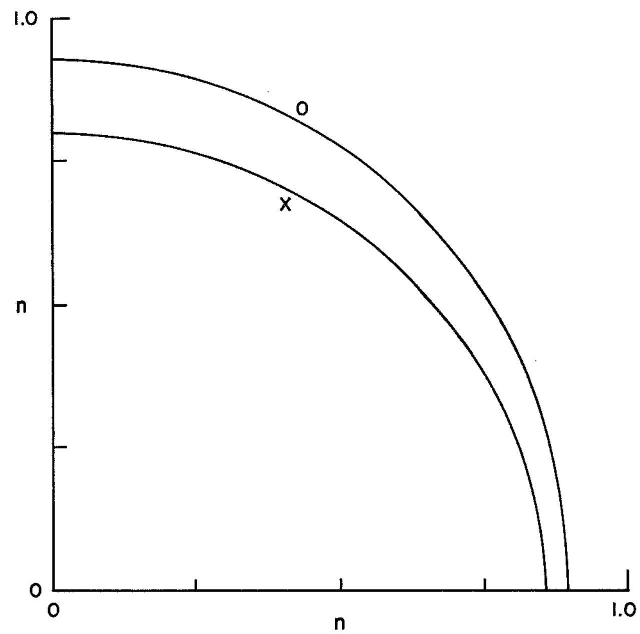
(d) extraordinary to extraordinary

Figure 3.4.3 The geometric factor G_m^i for forward scatter with X = 0.063 and Y = 0.075.

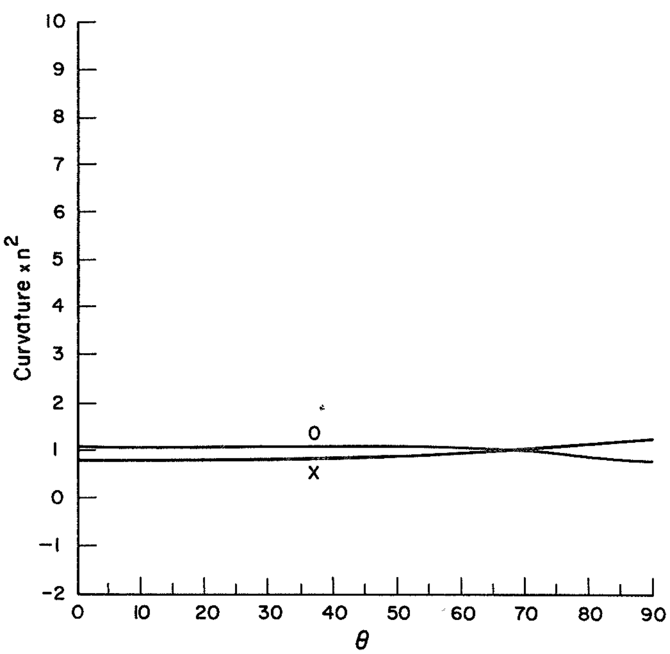
drops to zero at $\beta = 90°$. The range of angles for cross mode scattering is very small for this plasma. The peak value is slightly greater for scattering from "X" to "0" than from "0" to "X".

Figure 3.4.4 shows the refractive index and curvature for a plasma with $X = 0.2$ and $Y = 0.447$. X and Y are now much larger and so there is a much larger difference between the ordinary mode and the extraordinary mode. Figure 3.4.5 shows the geometric factor for this plasma. Sections (a) and (d) show the much larger range of values for scattering to the same mode for the larger X and Y here. In this case the extraordinary mode self scattering is always greater than that of the ordinary mode even at transverse propagation. This result is expected for the quasi-longitudinal part where the characteristic waves are elliptically polarized. The sense of rotation of the extraordinary mode is the same as the sense of rotation for the electron about the magnetic field. The cross mode scattering shown in sections (b) and (c) now has a much broader range and a peak value which is only about 2 orders of magnitude less than the self mode scattering. The peak value of the cross mode scattering is again slightly greater for "X" to "0" than for "0" to "X".

The two preceding examples are for very simple plasmas and the results are quite regular. This is not the case for those plasmas for which the shape of the



(a) refractive index n



(b) normalized curvature

Figure 3.4.4 The refractive index and Gaussian curvature for X = 0.2 and Y = 0.447.

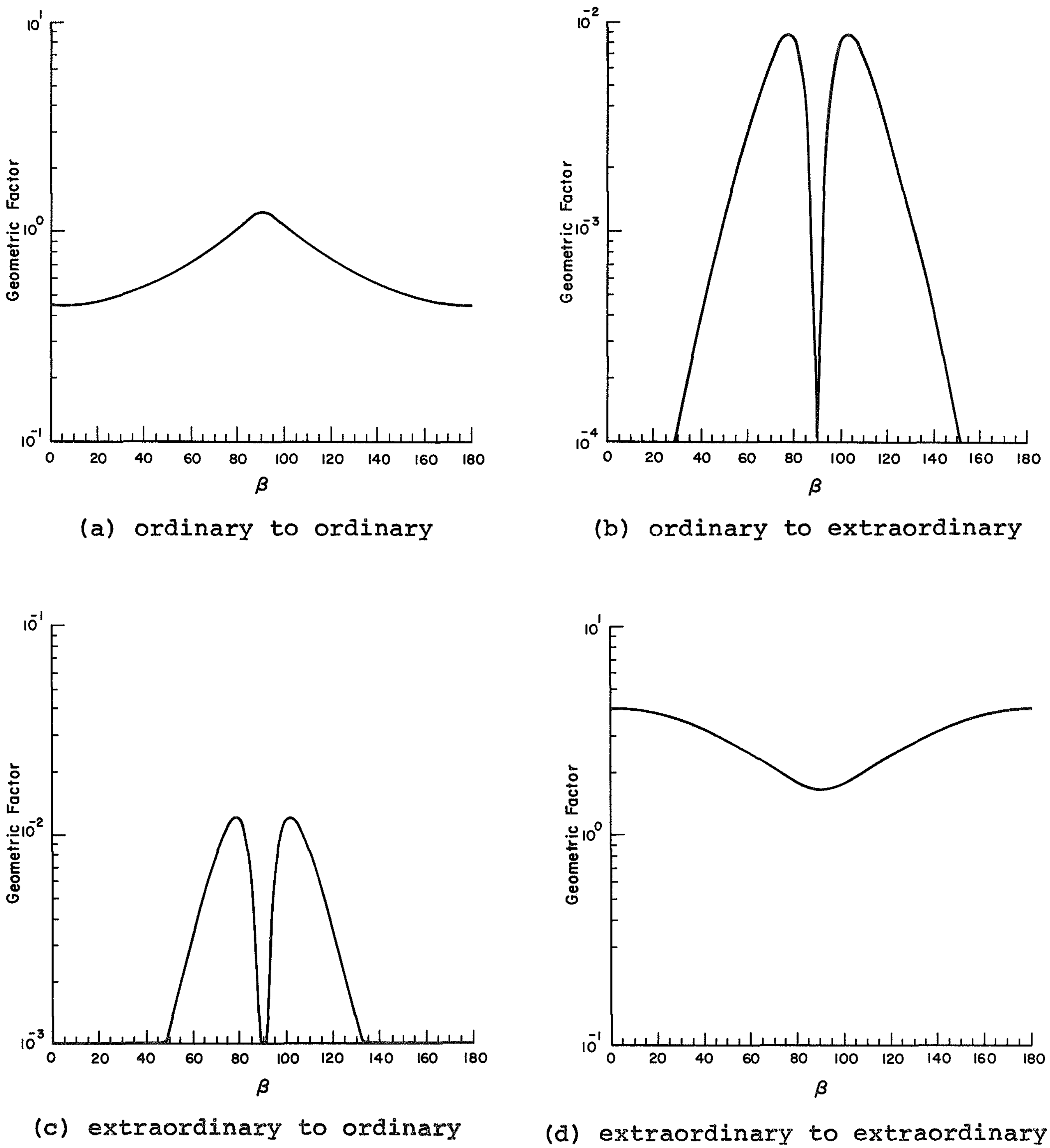
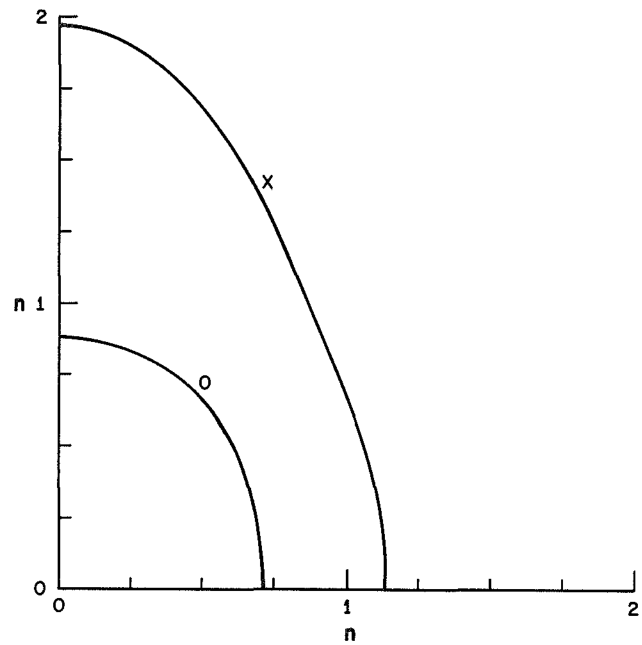


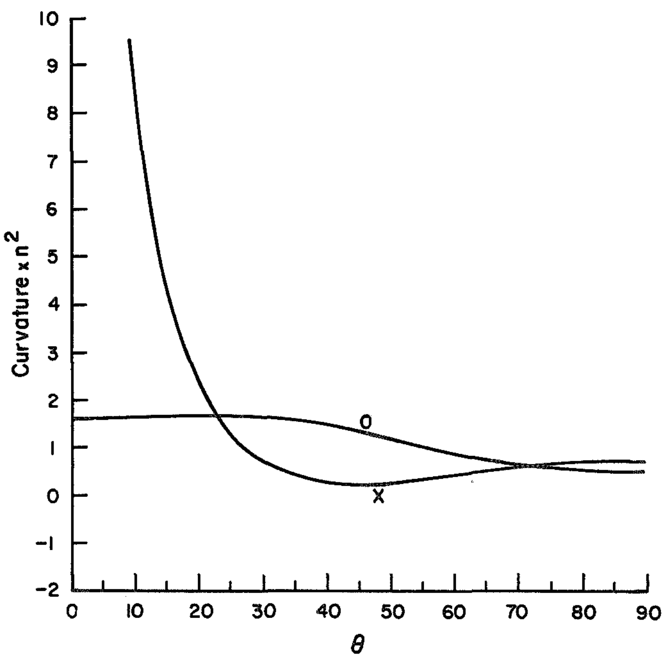
Figure 3.4.5 The geometric factor G_m^i for forward scatter with X = 0.2 and Y = 0.447.

refractive index surface of the extraordinary mode differs significantly from the ordinary mode. The next two examples are of this type of plasma.

Figure 3.4.6 shows the refractive index and the Gaussian curvature of a plasma with $X = 0.5$ and $Y = 1.173$. The major feature of the extraordinary refractive index surface is the large relatively flat area from about $\theta = 20°$ to $\theta = 60°$ for which ρ is approximately 65°. Figure 3.4.7 shows the geometric factor for forward scattering for this plasma. The ordinary to ordinary scattering shown in section (a) has the same form as the preceding plasmas with a larger range from longitudinal to transverse propagation. The cross mode scattering shown in sections (b) and (c) has the same general shape as before peaking at about 65°. The peak value of "0" to "X" scattering is now about an order of magnitude larger than "X" to "0" scattering. The cross mode scattering from ordinary to extraordinary is also of the same order of magnitude as the self mode scattering. The real complication for this plasma is in the self mode scattering for the extraordinary mode shown in section (d). The most salient feature here is the rapid changes in the amplitude around $\beta = 65°$. The reason for these rapid changes is that when $\beta \approx 62.3°$ $RQ \sin\theta = \cos\theta$ which causes the x component of equation A1.4 ($\gamma = 0$) to pass through zero from negative to positive. At this point $\vec{a}^*\cdot\overline{\overline{M}}\cdot\vec{a} = \frac{(2X-1)}{X}(1-Y^2)$. Since $X = .5$ for the

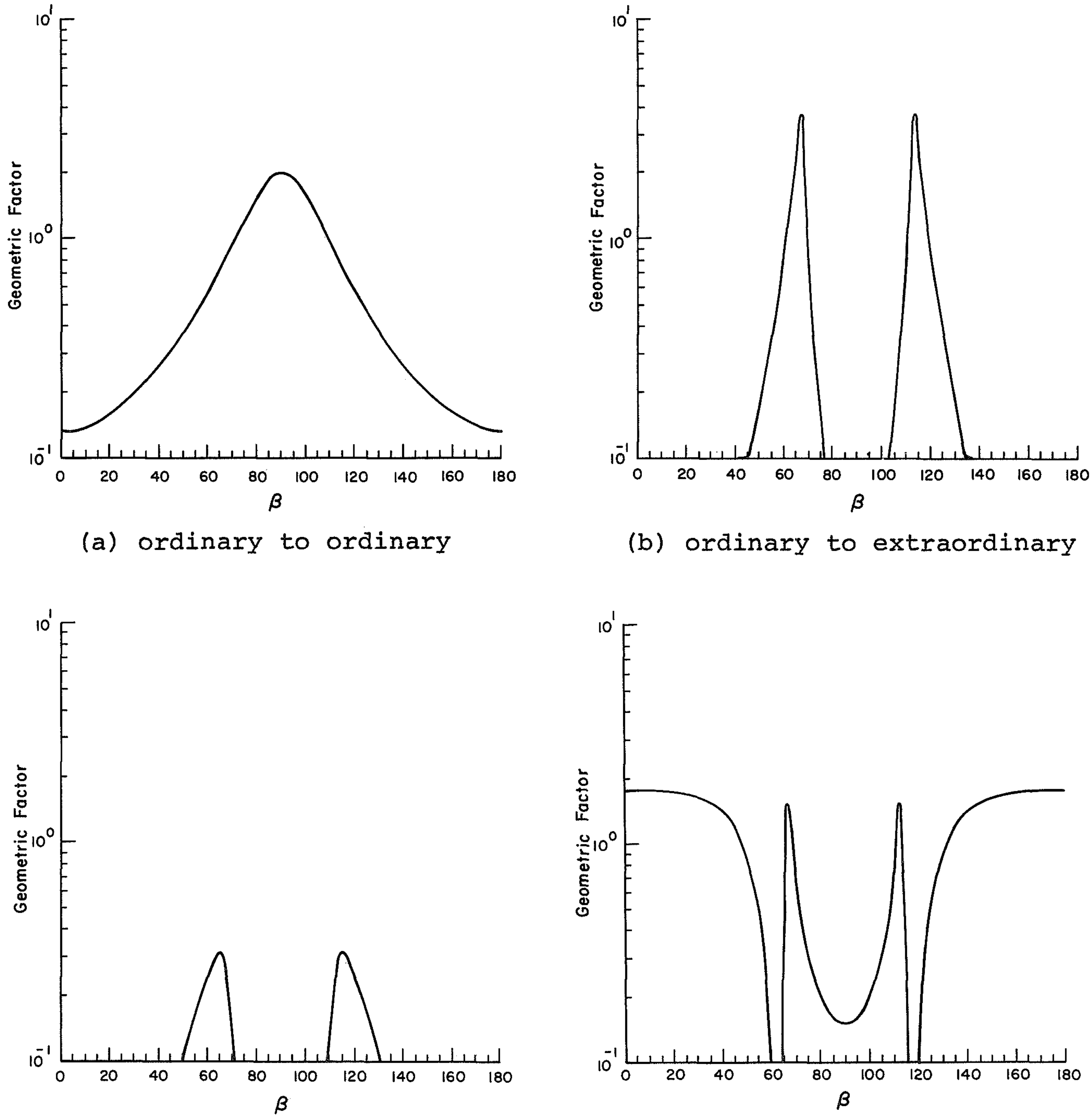


(a) refractive index n



(b) normalized curvature

Figure 3.4.6 The refractive index and Gaussian curvature for X = 0.5 and Y = 1.173.



(a) ordinary to ordinary

(b) ordinary to extraordinary

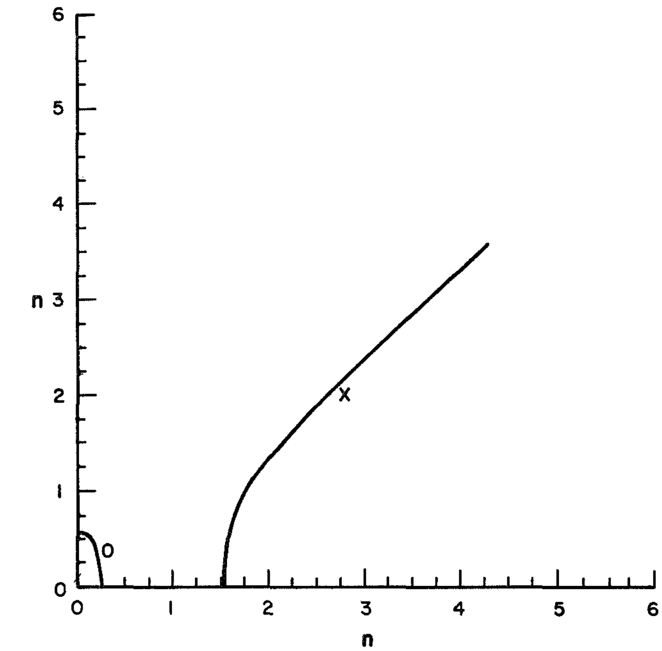
(c) extraordinary to ordinary

(d) extraordinary to extraordinary

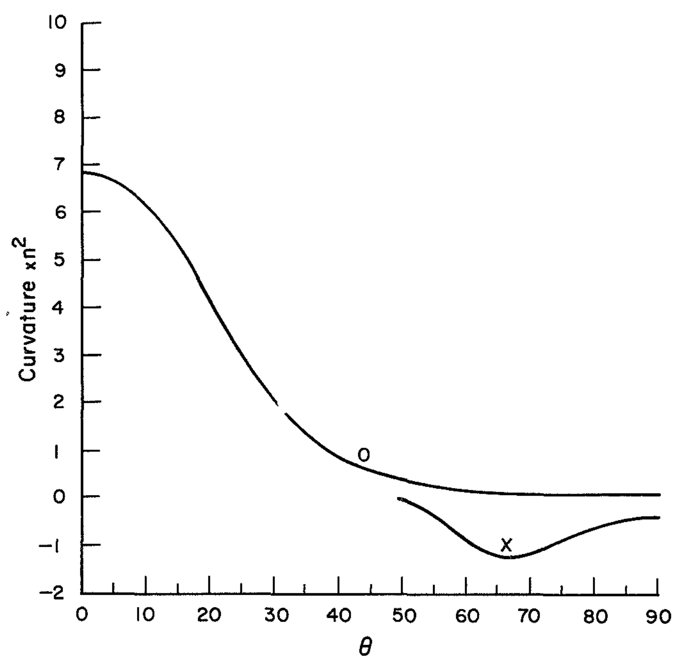
Figure 3.4.7 The geometric factor G_m^i for the forward scatter case with X = 0.5 and Y = 1.173.

case being considered here, the geometric factor goes to zero at this point. This type of behavior is always possible when $\frac{n^2}{1-X}$ is greater than one (RQ sin θ = cos θ $\rightarrow$ $\frac{n^2}{1-X}\sin^2\theta = 1$).

Figure 3.4.8 shows the refractive index and Gaussian curvature for a plasma with X = 0.925 and Y = 0.354. The major feature of the refractive index shown in section (a) is the asymptotic behavior of the extraordinary refractive index. The asymptote is at $\theta \approx 50°$ which has a ray direction of 135°. The curvature for the extraordinary mode starts at 50° because of this behavior. Figure 3.4.9 shows the geometric factor for this plasma. For ordinary self mode scattering shown in section (a) the shape of the curve is the same as before but the range of the curve is now from 8×10^{-2} at $\beta = 0$ to 1.33×10^{1} at $\beta = 90°$. This results in a beam about $\beta = 90°$. The scattering into the extraordinary mode from either mode, sections (b) and (d), is much different than anything encountered previously. These curves would go to infinity if the refractive index were allowed to do so, but the cold plasma theory used here is not valid at such large values of the refractive index. Actually a warm plasma theory would not go to infinity but the geometric factor would still be very large and very narrow in shape. The self mode factor is about 2 orders of magnitude larger than the ordinary mode factor. At $\beta = 90°$ the self mode factor for the extraordinary mode has a value of 5×10^{3}.



(a) refractive index n



(b) normalized curvature

Figure 3.4.8 The refractive index and Gaussian curvature for X = 0.925 and Y = 0.354,

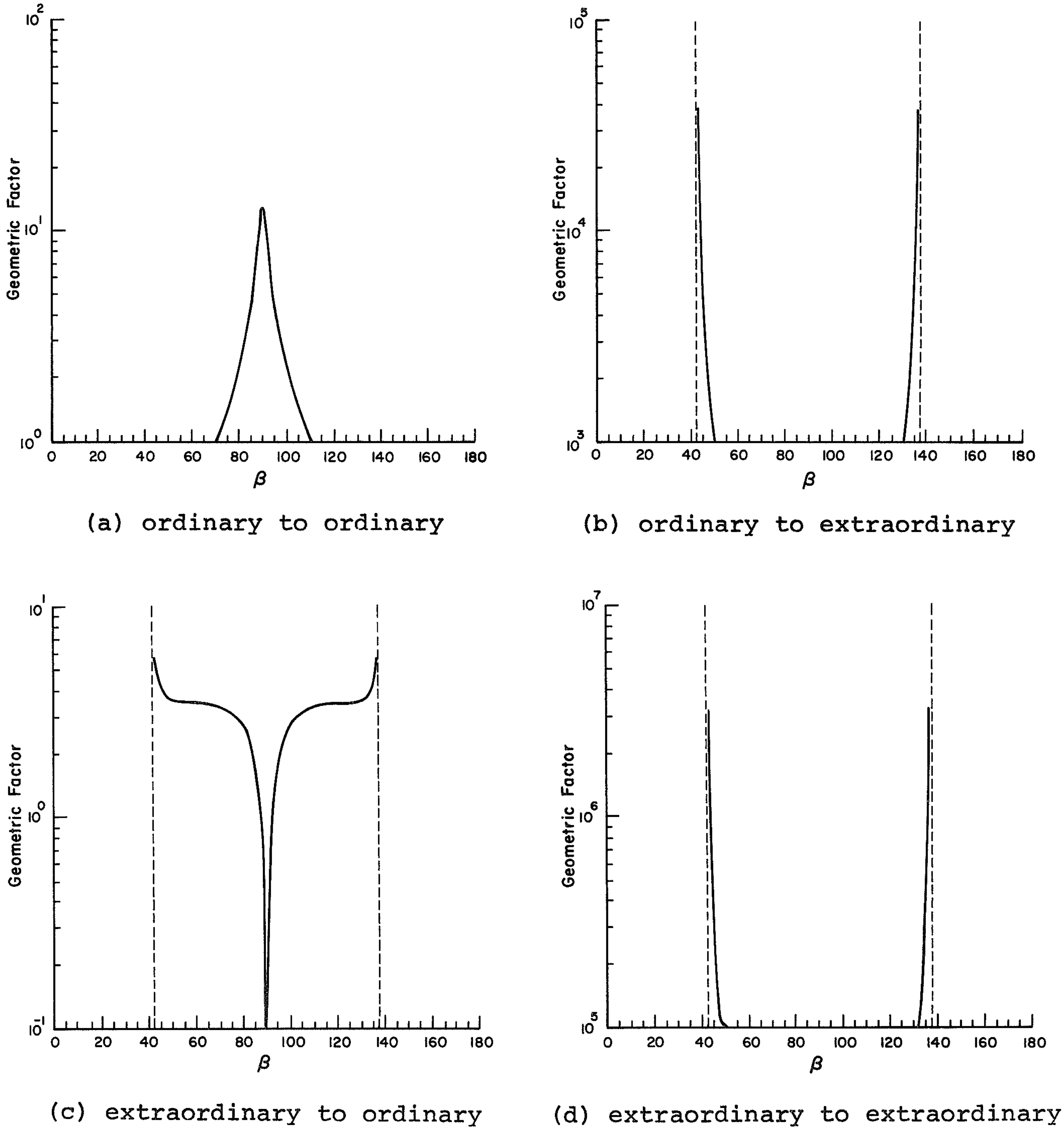


Figure 3.4.9 The geometric factor G_m^i for the forward scatter case with X = 0.925 and Y = 0.354.

These are just some representative examples for the geometric factor of the forward scatter case. For this case the geometric factor needs only three parameters to completely describe it, X, Y, and the angle the ray direction makes with the magnetic field. The general case requires five parameters to completely describe it, X, Y, the angles the incident and scattered ray make with the magnetic field and the azimuthal angle between the incident and scattered rays. This makes a meaningful display of G_s^i for the general case very difficult.

The best way to display the results is as a contour plot of the geometric factor for the natural physical situation. That is given X, Y and the incident ray, plot the geometric factor as a function of the two angles associated with the scattered ray. The geometry that is used in the figures is shown in Figure 3.4.10. The incident ray is always in the XZ plane and has the values $\phi_i = 0$, 45, 90 for each set of figures. The values of the contours are the values of the geometric factor G_s^i. Solid lines

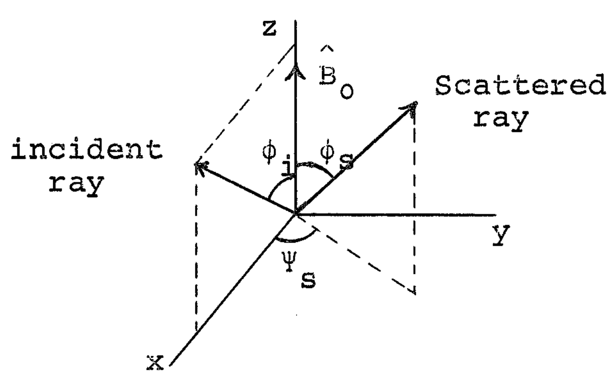


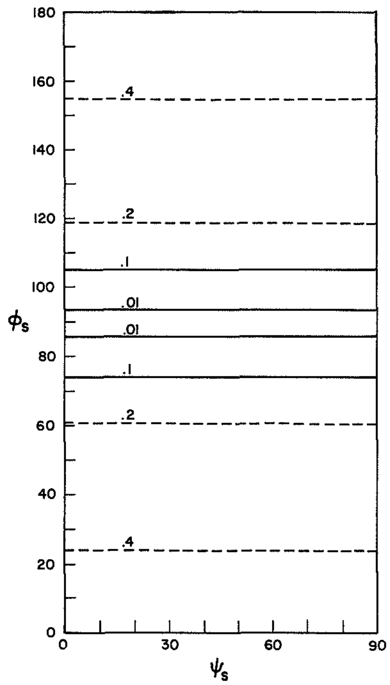
Figure 3.4.10 The geometry for the general case.

are powers of ten while dashed lines represent values other than 10^n. Both $\psi_s = 0$ and $90°$ are planes of even symmetry.

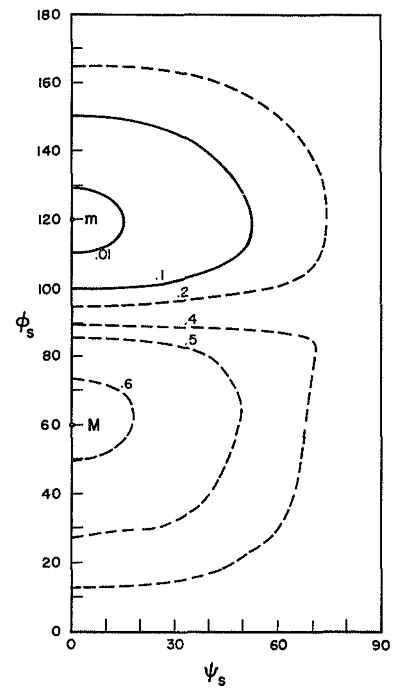
The first set of figures are for a plasma with $X = 0.2$ and $Y = 0.447$. The refractive index and the Gaussian curvature for this plasma are shown in Figure 3.4.4. The refractive index surfaces for this plasma are very regular and it is therefore typical of the type encountered in higher frequency studies of the ionosphere.

Figure 3.4.11 shows the geometric factor for scattering from ordinary to ordinary. For the incident ray along the magnetic field, G_s^i is independent of ψ_s as is seen by the straight lines in section (a). The same is almost true of the incident ray perpendicular to the magnetic field as shown by section (c). For $\phi_i = 0$ there is one minimum at $\phi_s = 90°$ (perpendicular to $\hat{B}_0$), and broad maxima in the direction of the magnetic field. For $\phi_i = 45°$ (section b) the maximum has shifted to $\phi_s = 60°$ for $\psi_s = 0$. As ψ_s increases to $90°$ G becomes very flat varying between .2 and .4. For $\phi_i = 90°$ there is one broad maximum at $\phi_s = 90°$ and minima along the magnetic field. The peak value of G varies by less than an order of magnitude from $\phi_i = 0°$ to $90°$.

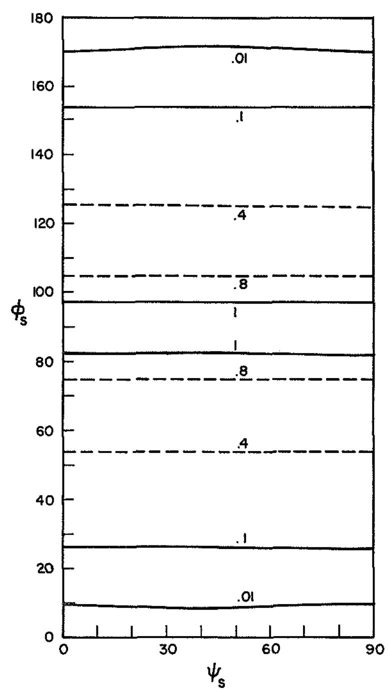
Figure 3.4.12 shows G for scattering from the ordinary ray to the extraordinary ray. The independence of ψ_s for $\phi_i = 0$ and $90°$ is again shown in sections (a)



(a) $\phi_i = 0.0$, $\Psi_i = 0.0$

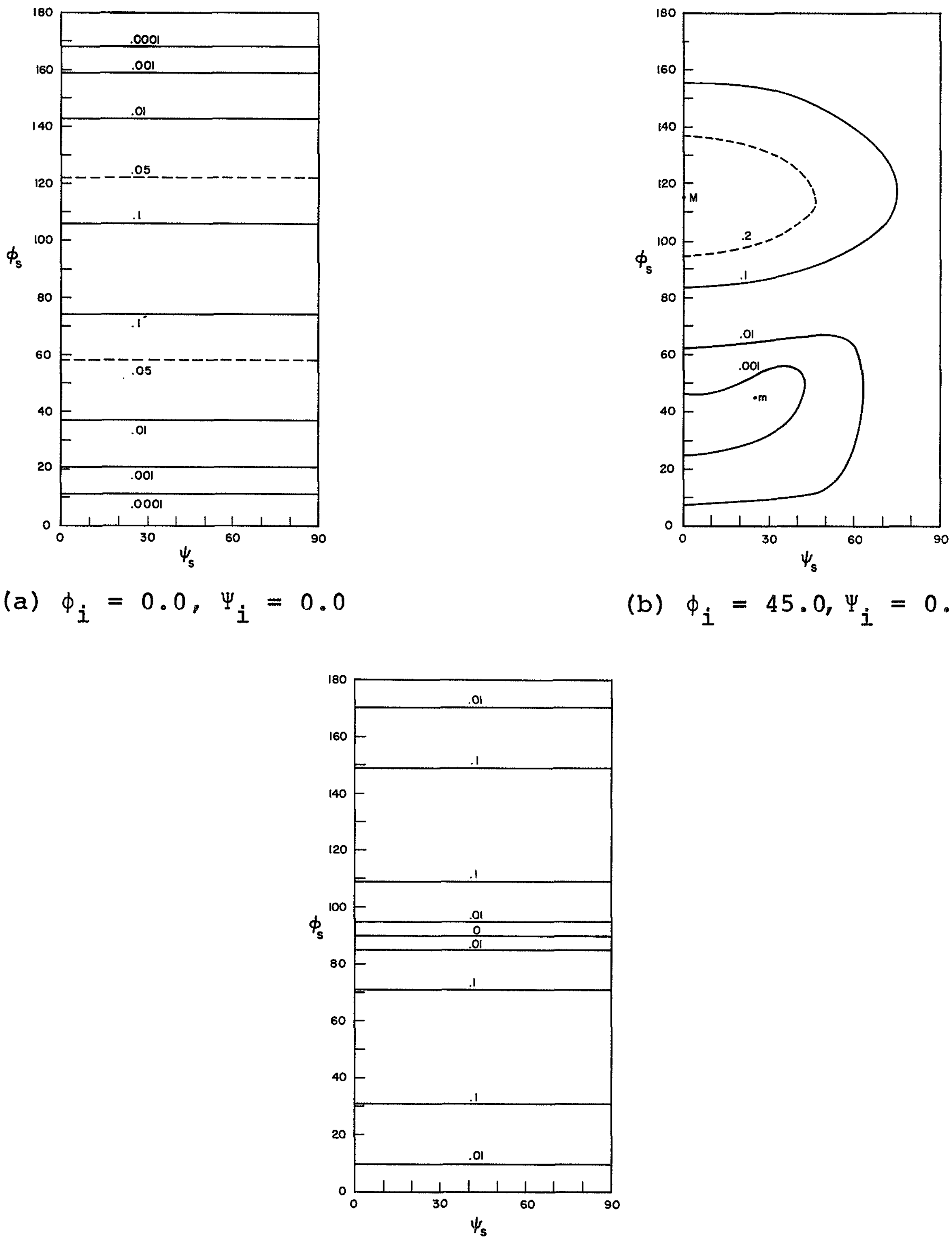


(b) $\phi_i = 45.0$, $\Psi_i = 0.0$



(c) $\phi_i = 90.0$, $\Psi_i = 0.0$

Figure 3.4.11 The geometric factor for ordinary to ordinary scattering with X = 0.2 and Y = 0.447.



(a) $\phi_i = 0.0$, $\Psi_i = 0.0$

(b) $\phi_i = 45.0$, $\Psi_i = 0.0$

(c) $\phi_i = 90.0$, $\Psi_i = 0.0$

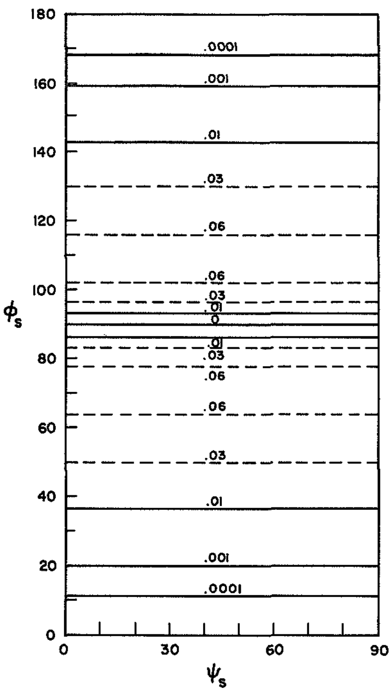
Figure 3.4.12 The geometric factor for ordinary to extra-ordinary scattering with X = 0.2 and Y = 0.447.

and (c), but now the peak values differ by about an order of magnitude. For $\phi_i = 0°$ there is a broad maximum at $\phi_s = 90°$ and minima along the magnetic field. For $\phi_i = 45°$ the maximum has moved to $\phi_s = 115°$ at $\psi_s = 0°$. The surface is not as flat in the region around $\psi_s = 90°$ as in the previous figure. For $\phi_i = 90°$ there are broad maxima at $\phi_s \approx 55°$ and $125°$ and minima at $\phi_s = 0, 90, 180°$.

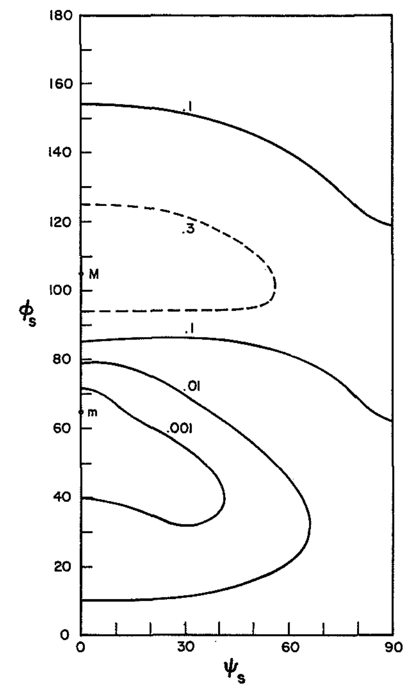
Figure 3.4.13 shows G for scattering from the extraordinary mode to the ordinary mode. Now only $\phi_i = 0°$ is independent of ψ_s. It has two maxima at $\phi_s \approx 70°$ and $110°$ and minima at $\phi_s = 0, 90, 180°$. For $\phi_s = 45°$ the shape of G is approximately the same as for "O"-"X" scattering. The peak value for $\phi_i = 45°$ is almost an order of magnitude greater than that of $\phi_i = 0°$. For $\phi_i = 90°$ G is no longer independent of ψ_s. With $\psi_s = 0°$ there is a sharp minimum at $\phi_s = 90°$ which increases in width as ψ_s approaches 90°. The peak value is about the same as that for $\phi_i = 45°$.

Figure 3.4.14 shows G for self mode scattering for the extraordinary mode. The major feature is that the surface is quite flat, compared to the other figures, for all values of ϕ_i, ϕ_s and ψ_s. As ϕ_i increases the range of values decreases a little.

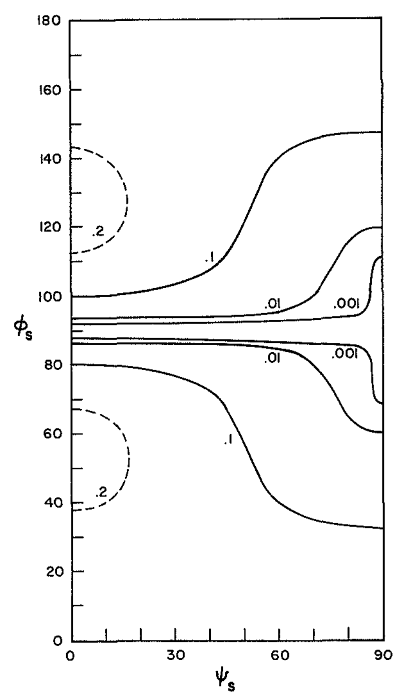
This set of figures shows that even for a refractive index as simple as this the value of the geometric factor is very variable. The exception to this is the self mode scattering of the extraordinary mode. This result is quite



(a) $\phi_i = 0.0,\ \Psi_i = 0.0$

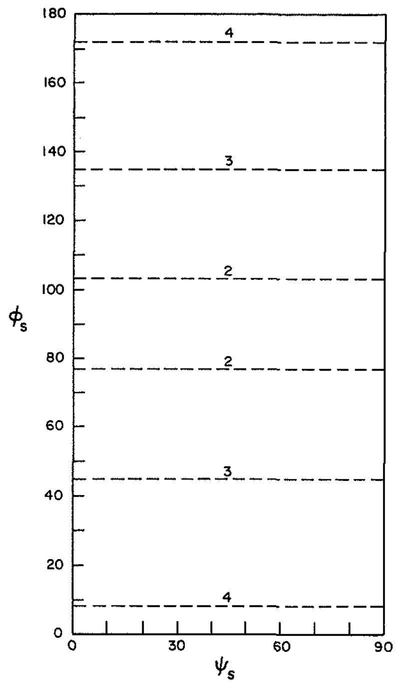


(b) $\phi_i = 45.0, \Psi_i = 0.0$

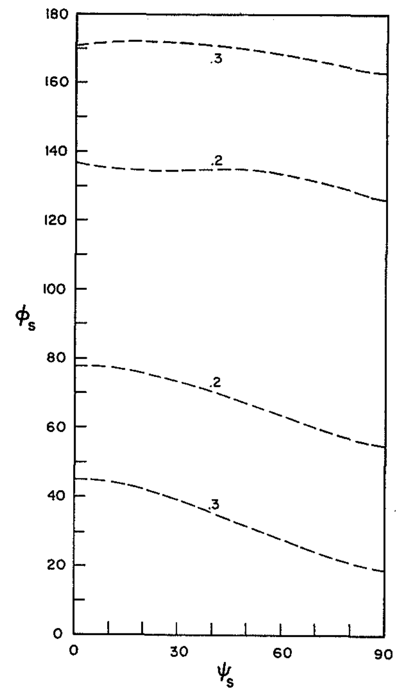


(c) $\phi_i = 90.0,\ \Psi_i = 0.0$

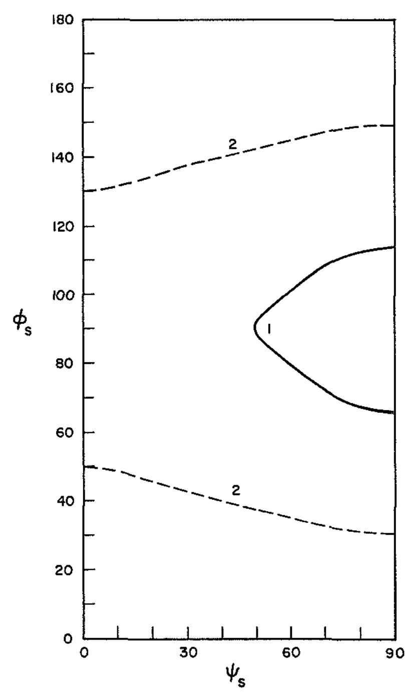
Figure 3.4.13 The geometric factor for extraordinary to ordinary scattering with X = 0.2 and Y = 0.447.



(a) $\phi_i = 0.0$, $\Psi_i = 0.0$



(b) $\phi_i = 45.0$, $\Psi_i = 0.0$



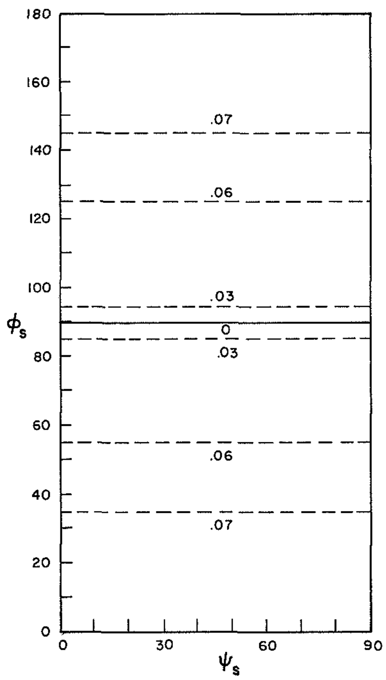
(c) $\phi_i = 90.0$, $\Psi_i = 0.0$

Figure 3.4.14 The geometric factor for extraordinary to extraordinary scattering with X = 0.2 and Y = 0.447.

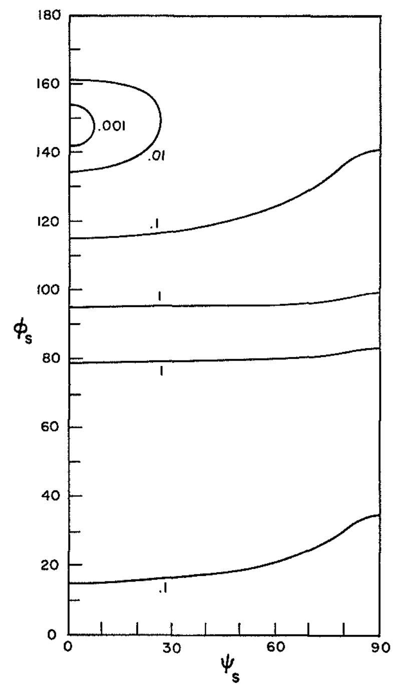
unexpected. Appendix A2 contains a series of plots of G for $\psi_s = 0°$ which give a better idea of the transition from $\phi_i = 0°$ to $\phi_i = 90°$.

If a more variable refractive index is chosen the results are quite different for most cases. Figure 3.4.8 shows the refractive index and Gaussian curvature for a plasma with X = 0.925 and Y = 0.354. The ordinary refractive index is similar to the refractive index of the preceding example but of smaller size. Therefore there should be some similarity for the ordinary mode self scattering. The extraordinary refractive index is completely different from the preceding case. The curvature is of the opposite sign and the refractive index has an asymptote at 49 degrees with respect to the magnetic field for the phase direction. The ray direction which corresponds to this is about 137°. Therefore the ray direction can vary between 42 and 137 degrees. Care must be taken in interpreting the results near the asymptotes since the cold plasma theory breaks down in this region and the Gaussian curvature is small and approaching zero which will violate the requirement that the third derivative of the surface be negligible compared to the second derivative (Gaussian curvature).

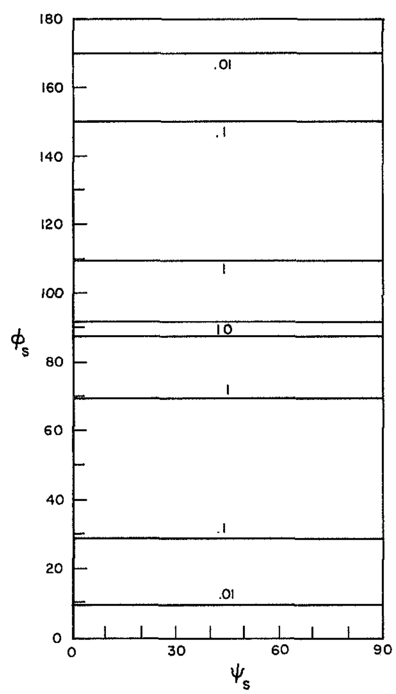
Figure 3.4.15 shows the self mode surface of the ordinary mode for this plasma. The shape of these curves are similar to those of the preceding case but the



(a) $\phi_i = 0.0$, $\Psi_i = 0.0$



(b) $\phi_i = 45.0$, $\Psi_i = 0.0$



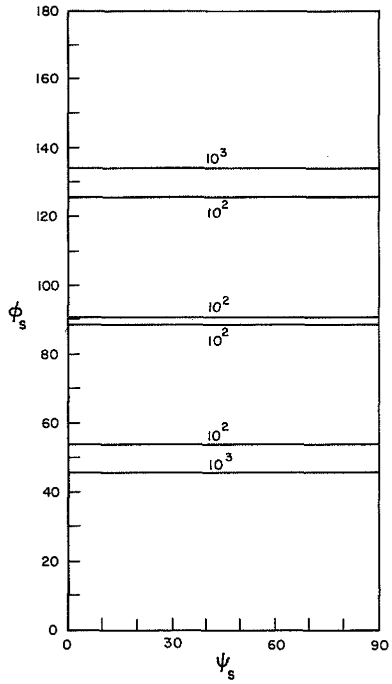
(c) $\phi_i = 90.0$, $\Psi_i = 0.0$

Figure 3.4.15 The geometric factor for ordinary to ordinary scattering with X = 0.925 and Y = 0.354.

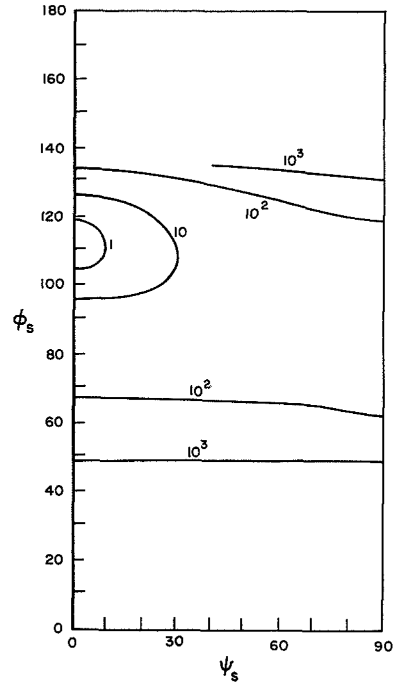
magnitude is different. The magnitude is about half of the preceding case for $\phi_i = 0°$ while it is an order of magnitude greater for $\phi_i = 90°$. For $\phi_i = 0°$ the maximum is very broad and flat decreasing very rapidly to a minimum at $\phi_s = 90°$. There is an order of magnitude increase from $\phi_i = 0°$ to $\phi_i = 45°$ and again from $\phi_i = 45°$ to $\phi_i = 90°$.

Figure 3.4.16 shows the cross mode scattering from ordinary to extraordinary. There are no values below $\phi_s = 45°$ or above $\phi_s = 135°$ for the reason mentioned above. The obvious feature of this figure is the very large values of G. If a more accurate theory were used in computing G the peak values might be a little less but these figures still show the trend of the surface for this condition. Of course there is no similarity to the corresponding surfaces of the preceding case. For $\phi_i = 0°$ there are minima at about $\phi_s = 75°$ and 105°, with very large values at $\phi_s = 45°$ and 135°. At $\phi_i = 45°$ the minimum is at $\phi_s = 115°$ and is deeper. Finally at $\phi_i = 90°$ the surface goes to zero at $\phi_s = 90°$.

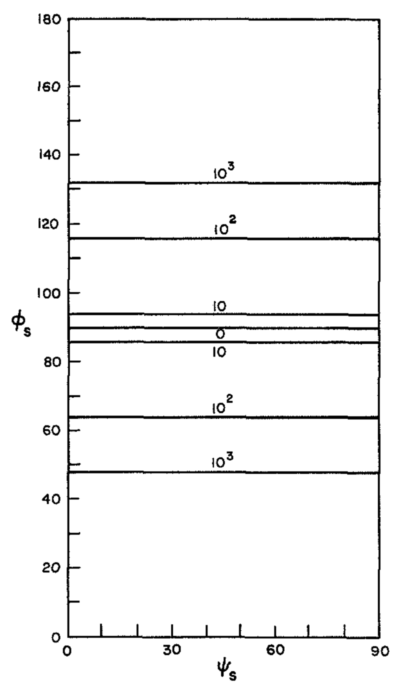
Figure 3.4.17 shows the cross mode surface from extraordinary to ordinary. There is no extraordinary mode for $\phi_i < 42°$ so there are only two sections to this figure. For $\phi_i = 45°$ there is a maximum at $\phi_s = 90°$ and a minimum at $\phi_s = 150°$. The peak value is about an order of magnitude greater than the previous case. At $\Psi_i = 90°$ G has a broad



(a) $\phi_i = 0.0$, $\Psi_i = 0.0$

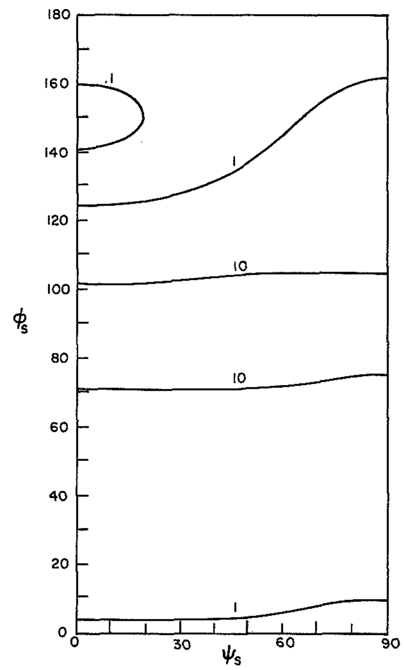


(b) $\phi_i = 45.0$, $\Psi_i = 0.0$

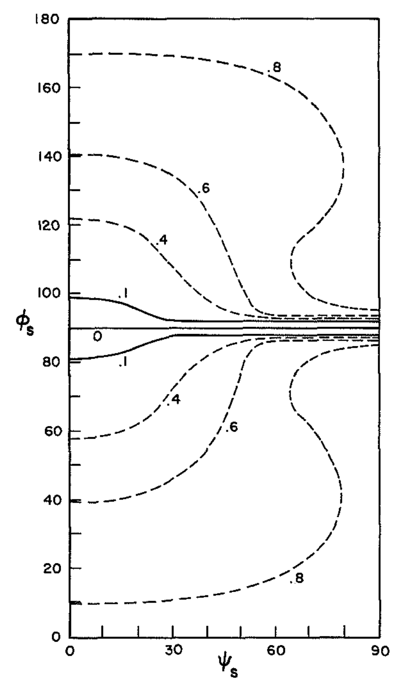


(c) $\phi_i = 90.0$, $\Psi_i = 0.0$

Figure 3.4.16 The geometric factor for ordinary to extraordinary scattering with X = 0.925 and Y = 0.354.



(a) $\phi_i = 45.0$, $\Psi_i = 0.0$



(b) $\phi_i = 90.0$, $\Psi_i = 0.0$

Figure 3.4.17 The geometric factor for extraordinary to ordinary scattering with X = 0.925 and Y = 0.354.

maximum which gets flatter as ψ_s increases and a sharp minimum at $\phi_s = 90°$.

Figure 3.4.18 shows the self mode surface for the extraordinary mode. The large values characteristic of scattering to the extraordinary mode are again evident and the same comments apply to them. The minima of both sections occurs at $\phi_s = 90°$ and $\psi_s = 90°$.

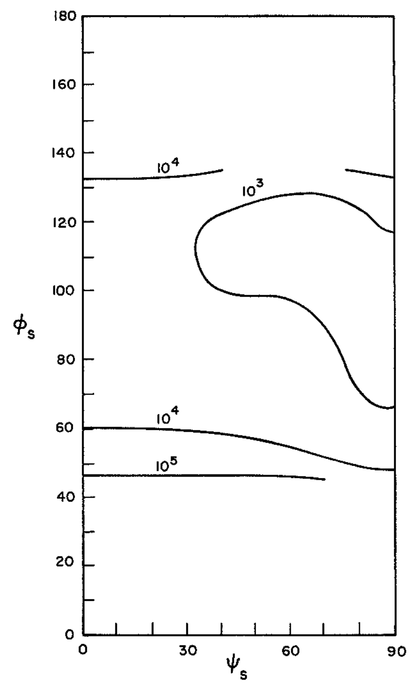
This last set of figures shows a markedly different behavior from the first set. The main reason for this behavior is the rather flat refractive index associated with the asymptotes. In this region the curvature is less than one and so G is increased from the value for the isotropic medium. Physically the large scattering associated with this flat area is expected since as the refractive index becomes flat, many more rays carry energy in approximately the same direction as the exact ray; that is, the rays diverge less, and so more energy should be received for a given incident power.

3.5 The Statistical Factor S_s^i

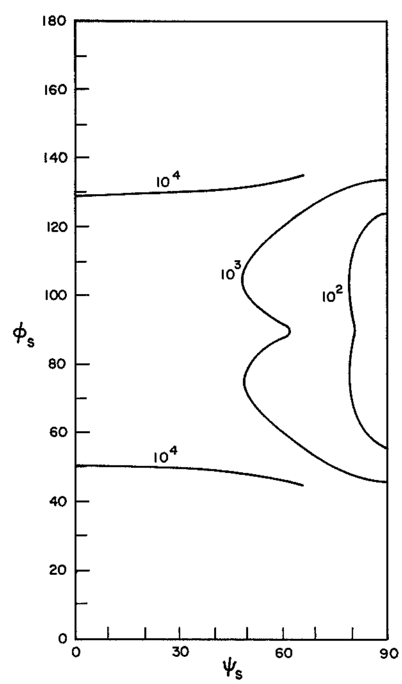
The statistical factor is given by equation 3.3.7 as

$$S_s^i = \langle(\Delta X)^2\rangle \; \Phi_B(\vec{k}_s - \vec{k}_i) \qquad 3.5.1$$

where Φ_B is the spectrum of the correlation function B for the fluctuations ΔX. This function depends upon all the previous parameters X, Y and the geometry of the magnetic



(a) $\phi_i = 45.0$, $\Psi_i = 0.0$



(b) $\phi_i = 90.0$, $\Psi_i = 0.0$

Figure 3.4.18 The geometric factor for extraordinary to extraordinary scattering for X = 0.925 and Y = 0.354.

field, the incident ray and the scattered ray, and in addition it depends upon the statistics of the fluctuations.

The reduction of this function to the isotropic case is straight forward. For the isotropic case $k_s = k_i = k$, so that equation 3.5.1 becomes

$$S = \langle(\Delta X)^2\rangle \ \Phi_B[k(\hat{r}_s - \hat{r}_i)] \qquad 3.5.2$$

where $\hat{r}_s$ is a unit vector in the direction of the scattered ray and $\hat{r}_i$ is a unit vector in the direction of the incident ray. This is exactly the factor P in equation 17 of Booker (1956). Thus equation 3.5.1 is just the obvious generalization of equation 3.5.2 to the anisotropic case considered here. Therefore all the previous theory developed for the isotropic case is valid here if it is evaluated at the more correct point $(\vec{k}_s - \vec{k}_i)$ instead of $k(\hat{r}_s - \hat{r}_i)$.

The forward scatter case is different from the back scatter case for S_s^i, which was not true for G_s^i. For the forward scatter case the incident ray and the scattered ray are in the same direction, so equation 3.5.2 is $\langle(\Delta X)^2\rangle \ \Phi_B(0)$. The same is true of equation 3.5.1 if $s = i$. Thus for plasmas with one saddle point for each mode the self mode forward scatter statistical factor is identical to that of the isotropic case. The cross mode factor is the addition given by the anisotropic situation.

If the correlation function of the fluctuations is a Gaussian function given by

$$B(\vec{R}) = \exp[-(\frac{X}{\ell_x^2} + \frac{Y}{\ell_y^2} + \frac{Z}{\ell_z^2})] \qquad 3.5.3$$

where ℓ is the correlation length, then the spectrum is

$$\Phi_B(\vec{k}) = \pi^{3/2}\, \ell_x \ell_y\, \ell_z \exp[-\frac{1}{4}(\ell_x^2 k_x^2 + \ell_y^2 k_y^2 + \ell_z^2 k_z^2)] \quad 3.5.4$$

Since $k = k_0\, n$ and $k_0 = \frac{2\pi}{\lambda}$, this equation becomes

$$\Phi_B(\vec{k}) = \pi^{3/2}\, \ell_x\, \ell_y\, \ell_z \exp[-\pi^2(\frac{\ell_x^2}{\lambda^2} n_x^2 + \frac{\ell_y^2}{\lambda^2} n_y^2 + \frac{\ell_z^2}{\lambda^2} n_z^2)] \qquad 3.5.5$$

Since n is not a very large number, the value of Φ_B will be determined primarily by the ratio of the correlation length to the free space wavelength. If this ratio is large (say 10) and n is of the order 1 then Φ_B will be negligibly small. This implies that the cross mode forward scattering is extremely weak when the ratio is large because Φ_B is small if n is of order 1 or if n is small the two modes are nearly identical for which the geometric factor discussed in the previous section is small. The case of n small is discussed further later in this section.

For the back scatter case the scattered ray is in the opposite direction from the incident ray so equation 3.5.2 becomes $\langle(\Delta X)^2\rangle\ \Phi_B[2k\ \hat{r}_i]$. For this case even the isotropic spectrum depends upon the ratio of the correlation length to the wave length. For this case

$$\Phi_B(\vec{k}_s-\vec{k}_i) = \Phi_B[-k_0(n_s+n_i)\hat{n}_i]$$

where $n_s n_i$ and $\hat{n}_i$ are evaluated for the ray direction $\hat{r}_i$. Therefore, if the isotropic case has a finite value for Φ_B, the anisotropic case will have finite solutions for Φ_B from both the self mode terms and the cross mode terms. In other words, if there is some back scattered power, the cross mode terms may be significant.

In the general case for the isotropic medium the spectrum is evaluated in the so called mirror direction $(\hat{r}_s-\hat{r}_i)$. That is, if a mirror were placed in the scattering volume such that its normal is in the direction $(\hat{r}_s-\hat{r}_i)$, the scattering problem is duplicated by the reflection problem. In the general case for the anisotropic medium the direction of evaluation does not have this simple meaning. The mirror direction is properly defined by the vector $\vec{k}_s-\vec{k}_i$ not by the vector $\hat{r}_s-\hat{r}_i$. This can be seen from Figure 3.5.1. If a wave is incident

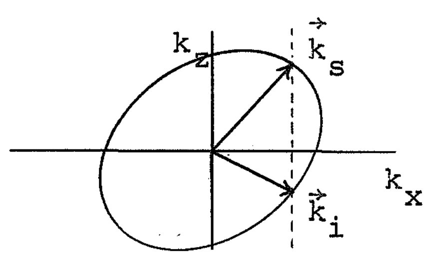


Figure 3.5.1 $\vec{k}$ in the case of a reflection.

on a perfect conductor in the X-Y plane with the wave vector $\vec{k}_i$ in the X-Z plane, then the reflected wave will have a wave vector $\vec{k}_s$ in the X-Z plane such that the X component of $\vec{k}_i$ and $\vec{k}_s$ are equal. The value of the z component is such that $\vec{k}_s$ lies on the refractive index surface. Therefore in this coordinate system the mirror vector $\vec{k}_s-\vec{k}_i$ is a vector in the z direction which is normal to the mirror. The same description holds for the isotropic medium but for this case $\vec{k}_s-\vec{k}_i = k(\hat{r}_s-\hat{r}_i)$ since the ray direction and the phase direction are the same. In the quantum mechanical scattering a term $\vec{k}_s-\vec{k}_i$ appears, which represents the change in momentum of the particle which occurs during the scattering process. This is just the conservation of momentum. Since the wave vector $\vec{k}$ is the wave analog of the momentum, this is an alternate description of the mirror vector. Therefore the scattering problem and the reflection problem are still the same as far as the mirror vector is concerned if it is properly defined.

An examination of $\Phi_B(\vec{k}_s-\vec{k}_i)$ would seem to indicate that the cross mode scattering should get better as the medium becomes more isotropic since k_s approaches k_i, but as was shown in section 4, the geometric factor for cross mode scattering decreases to zero as the medium becomes isotropic. Thus although Φ_B is increasing, its maximum value is only one, while the geometric factor for cross mode scattering is orders of magnitude less than that for self mode scattering. Therefore the cross mode power will be insignificant when compared to the self mode power.

So far nothing has been said about the other part of the statistical factor; $\langle(\Delta X)^2\rangle$. The mean squared value of a stationary random process is a measure of the intensity of deviations of the process from zero. In other words $\langle(\Delta X)^2\rangle$ is a measure of the intensity of the fluctuations. The appearance of this factor in the scattering cross-section says that the scattering cross-section is directly proportional to the intensity of the fluctuations.

Evaluation of S_s^i in specific cases requires, in addition to the information about the medium and the geometry, a specific model of the fluctuations which give rise to the scattering. Table 3.5.1 gives some specific correlation functions and the spectrum which corresponds

Table 3.5.1 Correlation Function and Spectrum Pairs

Type	Correlation Function	Spectrum
exponential	$e^{-\lvert R/\ell_0\rvert}$	$\dfrac{8\pi\ \ell_0^3}{[1+k^2\ell_0^2]^2}$
Gaussian	e^{-R^2/ℓ_0^2}	$\pi^{3/2}\ \ell_0^3\ e^{-(k\ell_0/2)^2}$
Bessel	$\dfrac{R}{\ell_0}\ K_1\left(\dfrac{R}{\ell_0}\right)$	$\dfrac{6\pi^2\ \ell_0^3}{[1+k^2\ell_0^2]^{5/2}}$
Cauchy	$\left[1+\dfrac{R^2}{\ell_0^2}\right]^{-2}$	$\pi^2\ \ell_0^3\ e^{-k_0\ \ell_0}$

to them (Wheelon, 1959). These pairs are evaluated for fluctuations which are isotropic. A discussion of the limitations, type of problems used in and the reasons for assuming each model can be found in Wheelon's paper. Tatarski (1961) also presents a good discussion of these different models for the fluctuations in section 4.4.

If the fluctuations are assumed to be anisotropic (having a different correlation length in different directions), the spectrum function is more complicated. Equation 3.5.3 is the appropriate Gaussian correlation function with different correlation lengths along the x, y and z-axis, and equation 3.5.4 is the spectrum function which corresponds to it.

The use of the Gaussian correlation function is not suggested by any physical attributes of the fluctuations. Its use is desired because it generally reduces the difficulty of the integrals involved since it goes to zero as the argument exceeds one or two correlation lengths.

3.6 The Approximations Used

The approximations used in obtaining the results of section 3.2 can be divided into two groups. The first group contains the approximations which are typical of problems of this type, no matter what type of medium is assumed. The second contains those approximations which are peculiar to the problem with an anisotropic background medium.

The first group contains three approximations. The first two are restrictions on the size of the scattering volume V. They are: the scattering volume must be large in terms of wavelengths and the scattering volume must be large in terms of correlation lengths. The third approximation is the far field approximation. This approximation can be stated more precisely as $|\vec{r}_{10}-\vec{r}| \approx r_{10}$ and $|\vec{r}_{20}-\vec{r}| \approx r_{20}$, where $\vec{r}$ ranges over the scattering volume. $|\vec{r}_0-\vec{r}|$ can be expanded as

$$|\vec{r}_0-\vec{r}| = r_0 \{1 - \frac{r}{r_0} \hat{r}_0 \cdot \hat{r} + \frac{r^2}{2r_0^2} [1 - (\hat{r}_0 \cdot \hat{r})^2] + \ldots\} \quad 3.6.1$$

Therefore if r/r_0 can be neglected in comparison with one the above approximation is satisfied.

The second group of approximations contains two approximations dealing with the parameters of an anisotropic medium. The first requires that the variation of the factors involved in the magnitude portion (d_s, $\vec{a}_s$, $\vec{a}_i$, k_s, k_i and C_s) is small when $\vec{r}$ varies over the scattering volume. The second approximation requires that the phase variations caused by the variation of $\vec{k}_s$ and $\vec{k}_i$ over the scattering volume are negligible. The second of these will be the most restrictive in general since phase variations generally have a far greater effect than amplitude variations.

The quantities involved in this second group of approximations are all explicit functions of the angle

θ and therefore their variation depends upon $\Delta\theta$, the variation of θ over the volume V. $\Delta\theta$ is related to $\Delta\rho$, the variation in the ray direction over the volume. $\Delta\rho$ is simply related to the distance to the scattering volume r_0 and the diameter of the scattering volume perpendicular to $\vec{r}_0$ designated by d_m. This relation is

$$\tan(\Delta\rho) = \frac{d_m}{2r_0}$$

If the far field approximation is met, $\Delta\rho$ is a small angle and therefore

$$\Delta\rho \approx \frac{d_m}{2r_0} \qquad 3.6.2$$

This equation connects the second group of approximations to the first group. Equation 3.6.2 is consistent with the far field approximation since the maximum value of r perpendicular to $\vec{r}_0$ is $d_m/2$.

Since $\Delta\theta$ is small it can be approximated by the first term in the Taylor series. Thus

$$\Delta\theta = \left.\frac{d\theta}{d\rho}\right|_{\theta_0} \Delta\rho \qquad 3.6.3$$

If equation 2.7.6 is used for $\frac{d\theta}{d\rho}$, this equation becomes

$$\Delta\theta = \frac{\sin \rho_0 \cos \alpha_0}{\sin \theta_0 \, n^2 \, C_n} \Delta\rho \qquad 3.6.4$$

where C_n is the Gaussian curvature of the refractive index surface given by n $(C = C_n/k_0^2)$. The factor

$\frac{\sin \rho_0 \cos\alpha_0}{\sin \theta_0}$ usually has a value much closer to one than n^2C_n for a given situation, so the main factor in determining $\Delta\theta$ is n^2C_n.

If n^2C_n is large, the effect of a given $\Delta\rho$ will be small. A large value of C_n means that the radius of curvature is small (see Figure 3.4.6) and therefore n is changing rapidly. Thus the more n changes for a given change in θ the smaller will be the effect of a given change in ray direction. These regions are the most advantageous from the standpoint of the second group of approximations since a given $\Delta\rho$ results in a smaller $\Delta\theta$ and therefore a smaller variation of the quantities in these approximations.

The opposite results occur when n^2C_n is small. Care must be taken here since if C_n is too small the third derivative of the refractive index will not be negligible and a more refined theory will be necessary. If the curvature is small the radius of curvature is large and the refractive index does not change much for a given change in θ. This is a "flat area" in the refractive index surface (see Figure 3.4.6 at about $\theta = 40°$). Now the effect of a given change in ρ is magnified instead of diminished. This is the worst case for the approximations since a given $\Delta\rho$ now results in a larger $\Delta\theta$ and therefore a larger variation of the quantities in these approximations.

Table 3.6.1 gives the variation of some of the quantities involved in the second group of approximations for the extraordinary mode of a plasma with X = 0.5 and

Table 3.6.1 Selected Variations of Quantities for the Extraordinary Mode with X = 0.5 and Y = 1.17

ρ	θ	n	n^2C_n	$\frac{d\theta}{d\rho}$	$\frac{1}{n}\frac{dn}{d\rho}$	$\frac{1}{n^2C_n}\frac{d(n^2C_n)}{d\rho}$
5°	1.11°	1.984	19.43	0.23	-0.035%	- 0.62%
50°	15.69°	1.779	3.98	0.59	-0.7 %	- 7.64%
65°	32.55°	1.446	0.52	2.72	-3.04 %	-32.12%

Y = 1.17 (see Figure 3.4.6). The discussion of the preceding paragraphs is shown in the columns labeled n^2C_n and $\frac{d\theta}{d\rho}$. In the first two cases n^2C_n is greater than one and so $\Delta\theta$ is less than $\Delta\rho$. In the last case n^2C_n is less than one and therefore $\Delta\theta$ is greater than $\Delta\rho$.

The sixth column is the variation of the phase. The phase is approximately given by

$$\vec{k}\cdot\vec{r} \approx \left(1 + \frac{1}{n}\frac{dn}{d\rho}\Delta\rho\right)\vec{k}\Big|_{r_0} \cdot \vec{r} \qquad 3.6.5$$

Thus $\frac{1}{n}\frac{dn}{d\rho}$ is the percentage change in phase for a unit change in ray direction. Table 3.6.1 shows that the phase variation can probably be ignored for the first two cases, but $\Delta\rho$ doesn't have to be very large before the phase variation of the last case becomes significant.

The last column gives the variation of one of the factors involved in the magnitude. Again the variation of the first two cases is probably negligible, but the last case has a rather large percentage variation and therefore may be significant. The remaining factors in the magnitude deal with the characteristic vector. The magnitude of the characteristic vector is a slowly varying function of θ and therefore its variation is always negligible. The remaining factor is the product of two characteristic vectors with $\overline{\overline{M}}$. For most cases this factor should have the same behavior as the magnitude of the characteristic vector. The exception to this will occur when the product $\vec{a}_s \cdot \overline{\overline{M}} \cdot \vec{a}_i$ has a small value.

The results of this section show that the additional approximations due to the anisotropic medium are consistent with the approximations normally encountered in this type of problem. At the same time they are highly dependent on the properties of the medium in the neighborhood of the saddle point and therefore generalizations can be misleading.

3.7 Applications of the Results

The results of this chapter will be applied to two problems. The first is a discussion of the modification of the theory used to interpret partial reflection data. The other problem is the possible ducting of whistlers along the magnetic field by the scattering process. The discussions of these topics will be qualitative instead of quantitative because of the need for a more refined theory, different in each case, to accurately describe the important features.

The partial reflection experiment is used as a means of measuring the electron density in the region 50-100 km (Belrose and Burke, 1964). The experiment consists of sending up a short pulse of one characteristic polarization and measuring the return echo and then doing the same for the other characteristic polarization. The ratio of the received power of the extraordinary mode to that of the ordinary mode is then related to the electron density through a theory involving an assumed collisional model. The frequency that is used in these experiments is around one megahertz which means that Y will be of significant value.

The theory that is used to determine the density is an application of the isotropic scattering theory of Booker (1956) to a plasma in which collisions are important and therefore there is loss. Since the scattering theory developed here is for a lossless plasma with negligible collision frequency, it cannot be applied directly to the partial reflection problem. Instead of redoing the theory to make it applicable, the partial reflection theory will be simplified so that it may be compared to the scattering theory developed here to determine what type of factors have been neglected by the isotropic scattering theory.

The theory of partial reflections is presented in a paper by Flood (1968). Equation 14 of that paper gives the ratio σ of the backscatter cross sections of the extraordinary mode to the ordinary mode for the quasi-longitudinal approximation as

$$\sigma_F = \frac{\sigma_x}{\sigma_o} = \frac{\Phi(\vec{q}_x)\ n_o^i\ \{1 - \exp[-(4\pi n_x^i/\lambda)C\tau]\}}{\Phi(\vec{q}_o)\ n_x^i\ \{1 - \exp[-(4\pi n_o^i/\lambda)C\tau]\}} \qquad 3.7.1$$

where

$\vec{q} = 2\vec{k} = 2k_0\ n^r\ \hat{k}_i$

n^i = imaginary part of the refractive index

n^r = real part of the refractive index

τ = radar pulse length

In this equation those terms that depend upon n^i are the result of absorption due to collisions and are not present in the simple plasma model used in the development of the scattering theory presented here.

The value of σ using the scattering cross section developed here (given by equation 3.3.5) for self-mode scattering is

$$\sigma = \frac{|\vec{a}_x^* \cdot \overline{\overline{M}} \cdot \vec{a}_x|^2}{|\vec{a}_o^* \cdot \overline{\overline{M}} \cdot \vec{a}_o|^2}\ \frac{n_o^2\ \sec^2\ \alpha_o\ |C_o|}{n_x^2\ \sec^2\ \alpha_x\ |C_x|}\ \frac{\Phi(\vec{q}_x)}{\Phi(\vec{q}_o)} \qquad 3.7.2$$

where the C used here is for the refractive index n. If the imaginary part of the refractive index is small, the real part will be essentially the same as for the lossless case. With this assumption, Flood's theory gives the modification to the lossless theory to account for loss. Therefore equation 3.7.2 serves as the correction to the portion of equation 3.7.1 to account for an anisotropic background medium. Using Flood's correction for loss, equation 3.7.2 becomes

$$\sigma = F\,\sigma_F \qquad 3.7.3(a)$$

where

$$F = \frac{|\vec{a}_x^{*}\cdot\overline{\overline{M}}\cdot\vec{a}_x|^2}{n_x^2\,\sec^2\alpha_x\,|C_x|}\;\frac{n_o^2\,\sec^2\alpha_o\,|C_o|}{|\vec{a}_o^{*}\cdot\overline{\overline{M}}\cdot\vec{a}_o|^2} \qquad 3.7.3(b)$$

F is just the ratio of the self-mode geometric factors for the extraordinary mode and ordinary mode. The factor F given by equation 3.7.3(b) is only approximate since it was derived using a collisionless plasma theory which is not valid in the region of the ionosphere of interest here. The modification of F should not be too great in so far as it depends upon the real part of the refractive index. Any additional factor in F due to the imaginary part of the refractive index would not be such that it would make F closer to unity, so F given by equation 3.7.3(b) should give a reasonable first approximation to the modification of Flood's theory due to the anisotropy of the background medium.

Due to the low frequencies used in the partial reflection experiment, Y will have a significant value and therefore F can be significantly different from one. Belrose and Burke (1964) used a frequency of 6.275 MHz for their experiments. This corresponds to a value of Y = 0.24 for the earth's magnetic field. For an electron density of $N = 10^4$ electrons/cc, X = 0.142. The value of F can be obtained for longitudinal propagation from Table 3.4.1. It is

$$F = \left[\frac{2(1-X)(1+Y)+XY}{2(1-X)(1-Y)-XY}\right]^2 \qquad 3.7.4$$

With the above values of X and Y, F = 1.6. Therefore even for these small values of X and Y, σ is underestimated by about 50% using the theory given by Flood. Since 6 MHz is a high frequency for a partial reflection experiment, the degree of error will be much worse for the frequencies usually used in this type of experiment.

This addition to the theory of partial reflections causes another complication. In the partial reflection experiment σ is measured from the data for different heights. At a low height the absorption is very small and so the density N can be determined. This N together with an assumed model for the collision frequency can be used to solve for the density N at a slightly higher height and so on. The new complication is that the factor F also depends upon the density and therefore makes the determination of the density more difficult.

In the partial reflection experiment only the self-mode scattering is considered. Additional information may be available if cross-mode scattering is considered. In section 3.5 the cross-mode scattering was shown to be comparable to the self-mode scattering for back scatter if there is a significant difference between the refractive index for the two modes.

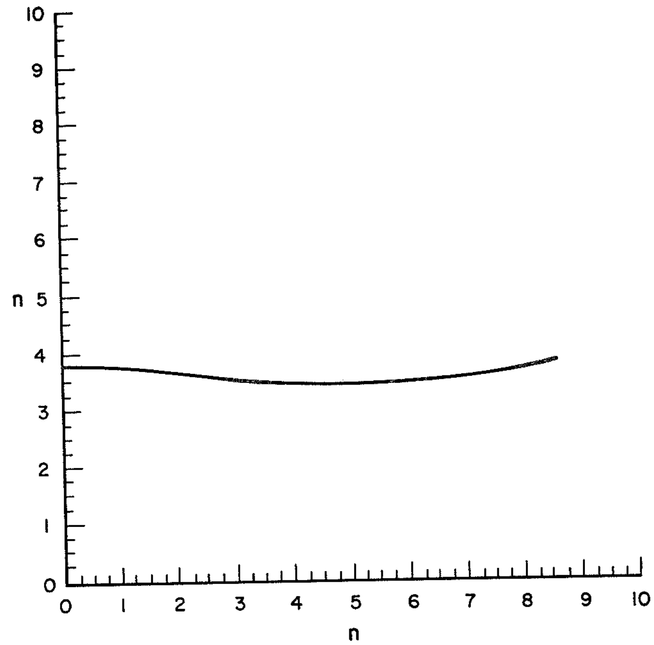
In order to do an accurate calculation for the type of plasma encountered in the partial reflection experiment the theory developed here must be redone from the start. A new Green's function must be derived along the line of the development of the generalized Appleton-Hartree formula given by Sen and Wyller (1960). Characteristic fields appropriate to this type of plasma must also be used. The rest of the theory should proceed smoothly but a new set of approximations will probably result.

The second application concerns whistler propagation. Whistlers are manifested through a mode of radio propagation in the upper ionosphere in which the energy propagates nearly along the magnetic field to the opposite hemisphere. The whistler may bounce back and forth between the two hemispheres several times. Whistlers originate from lightning flashes in the atmosphere. The name whistler comes from the fact that this mode is highly dispersive, so if a pulse starts out it will quickly be distorted with the higher frequency arriving first giving a descending tone if the signal is detected.

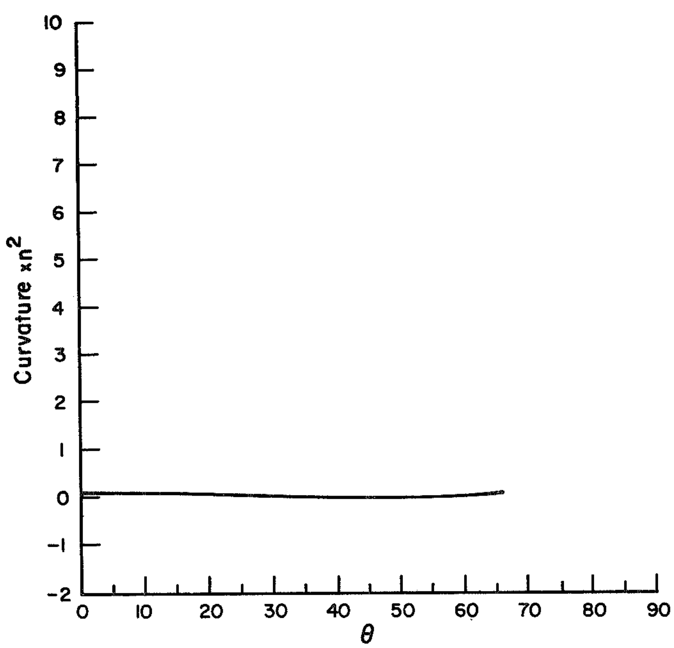
The mechanism by which the whistler is guided along the magnetic field line is not definitely known. Helliwell (1965) presents an extensive discussion on the ducting of whistlers by field-aligned irregularities. The application of the theory presented here will show that scattering in general in an anisotropic medium can cause the guiding of whistlers.

The theory of whistlers shows that the minimum time delay occurs for a frequency such that $Y = 4$. This is the so called "nose frequency." The discussion that follows is for this frequency.

Figure 3.7.1 shows the refractive index and normalized Gaussian curvature for a plasma with $X = 40$ and $Y = 4$. For this plasma only one mode propagates. The obvious feature of this refractive index surface is the very small range of the ray direction centered about zero degrees. The Gaussian curvature for this plasma (shown in section (b)) has a very small value and goes through zero twice. The first zero corresponds to a point of inflection with a ray direction of seven degrees and $\theta = 32$ degrees and the second corresponds to a point of inflection with a ray direction of zero degrees and $\theta = 55$ degrees. This second point of inflection is the important one for the



(a) refractive index n



(b) normalized curvature

Figure 3.7.1 The refractive index and Gaussian curvature for X = 40.0 and Y = 4.0.

propagation of whistlers. Since there are two points of inflection, the contribution to the field in a given direction may come from as many as three saddle points. The first saddle point extends from $\theta = 0°$ to $\theta = 32°$ with ray directions from $\rho = 0°$ to $\rho = 7°$. The second saddle point extends from $\theta = 32°$ to $\theta = 55°$ with $\rho = 7°$ to $\rho = 0°$. The third saddle point extends from $\theta = 55°$ to $\theta = 74°$ with $\rho = 0$ to $\rho = 343°$. The refractive index was terminated at a value of 10 for clarity of presentation.

Figure 3.7.2 shows the geometric factor for scattering from a wave incident along the magnetic field, which is zero degrees in the figure, versus the scattered ray angle. The incident mode in this figure is the first one for which $\theta = 0°$. Section a shows the self mode geometric factor. It goes to infinity at a ray angle of seven degrees. This is due to the point of inflection at $\theta = 32°$ which has zero Gaussian curvature. Near this point the Green's function used here is not valid and a more refined theory must be used as was mentioned in section 2.5 in connection with vanishing Gaussian curvature. Section b also shows this condition as well as a similar condition at $\theta = 55°$. At this point of inflection the ray angle is zero degrees. Section c shows the sharp rise at zero degrees and another at 17°. The latter one arises from the asymptotic behavior of the refractive index.

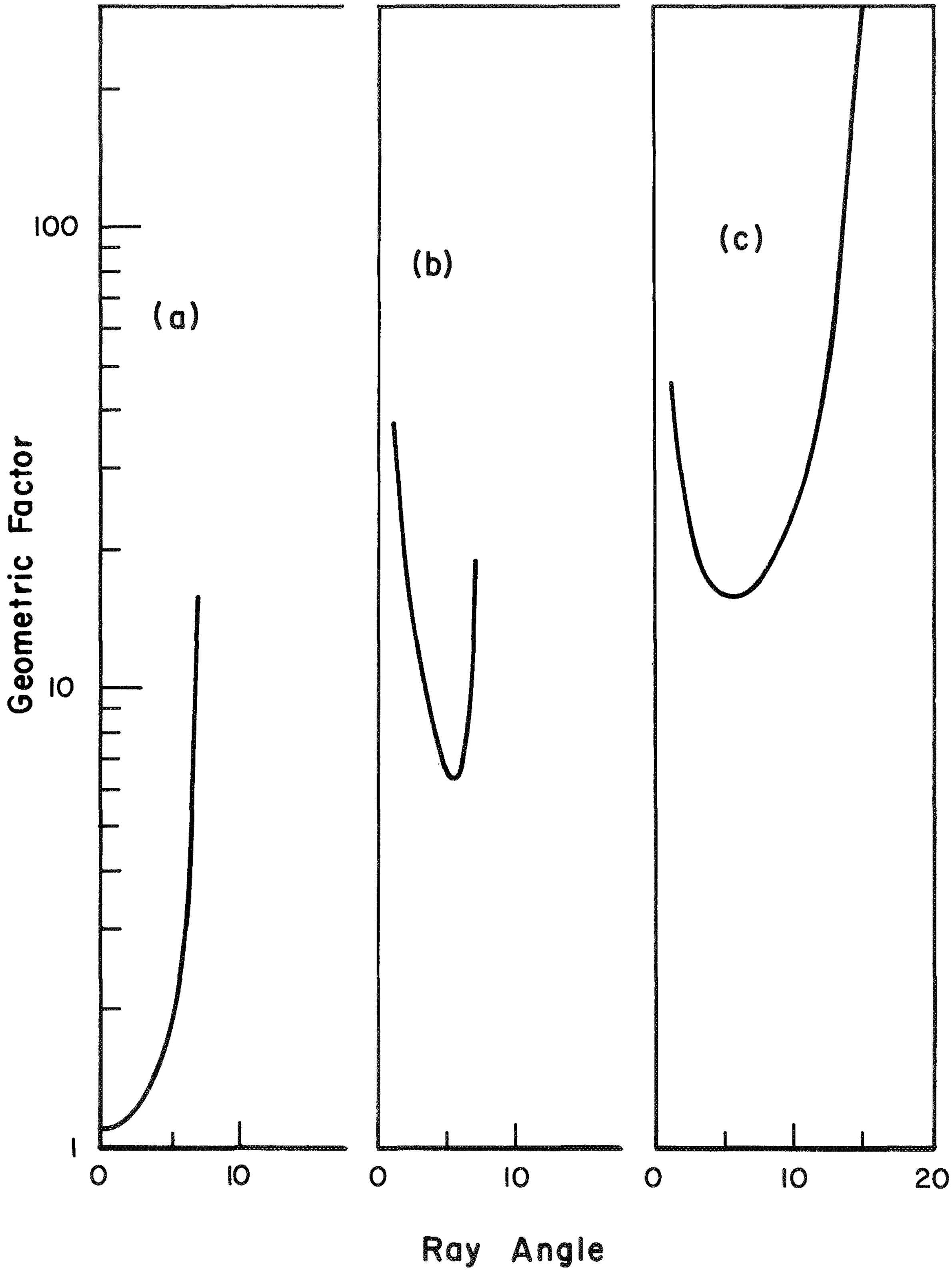


Figure 3.7.2 The geometric factor for a plasma with X = 40.0 and Y = 4.0.

(a) self mode

(b) second stationary point

(c) third stationary point

This figure indicates that there are three beams which guide the scattered energy. The magnitude of the beam near 17° is wrong since the cold plasma theory used here is invalid and a warm plasma theory has to be used. Even with the warm plasma model the fields from this beam will decrease as r^{-1}. The other two beams are associated with points of inflection and therefore, as was discussed in section 2.5 the fields will decrease as $r^{-5/6}$ which makes these two beams very important for large distances.

This work shows that there is a mode of propagation which will guide the energy along the magnetic field. If the same computations are made for incident rays inclined to the magnetic field the same beams appear with essentially the same values.

For $\rho = 0$ there is another stationary point given by $\theta = 55°$. The above discussion shows that the curves for this wave have the same shape as Figure 3.7.2 and approximately the same magnitude. Therefore that figure can be used for this wave as well but now section b is the self mode term. Thus the self mode term has a beam along the magnetic field.

For the forward scatter case the spectrum is given by

$$\Phi(\vec{k}_s - \vec{k}_i) = \Phi[k_0 \; (n_s - n_i)\hat{k}_i] \qquad 3.7.5$$

where the s and i refer to stationary points only since there is only one mode. If s = i, Φ is a maximum and

and equal to 1. Therefore the scattering cross section depends only on the geometric factor and the mean square value of the fluctuations but not on the correlation length of the fluctuations. For the cross mode terms ($s \neq i$) the scattering cross section also depends very strongly on the ratio of the correlation length to the wavelength in the medium. Therefore the cross mode terms depend upon the average size of the irregularities doing the scattering.

The above discussion indicates that scattering from irregularities may be a mechanism for guiding whistlers along the field lines. A more accurate theory would have to be used in order to obtain meaningful numbers for the beam directions $\rho = 0°$ and $\rho = 7°$. This theory would have to account for the case of vanishing Gaussian curvature by using the third derivative of the refractive index surface in the derivation of the Green's dyadic. If results for the beam at $\rho = 17°$ are desired a warm plasma model must be used.

CHAPTER IV

THE RADIO STAR PROBLEM

4.1 Introduction

The radio star problem has a geometry essentially the same as the forward scatter problem considered in Chapter III, with an infinite slab replacing the scattering volume. The transmitter is located far above the slab and the receiver is located far below the slab so that the signal propagates through the slab of irregularities. The longitudinal correlation of the electric field is the quantity of interest in this chapter. In this problem there is no restriction on the size of the antenna beam for either the transmitter or the receiver.

This type of problem occurs in the reception of signals from earth satellites or radio stars on the surface of the earth. It may also have applications in laboratory plasma diagnosis.

4.2 The Longitudinal Correlation of the Electric Field

The geometry for the radio star problem is shown in figure 4.2.1. A transmitter T is located at the origin 0 of the coordinate system. A characteristic wave with a group velocity in the direction $\hat{r}'$ is scattered from a point $\vec{r}'$ and travels to the receiver R, located at $\vec{r}$, with a group velocity in the direction $\hat{R}$. The unscattered electric field has a group velocity in the direction $\hat{r}$. The slab of

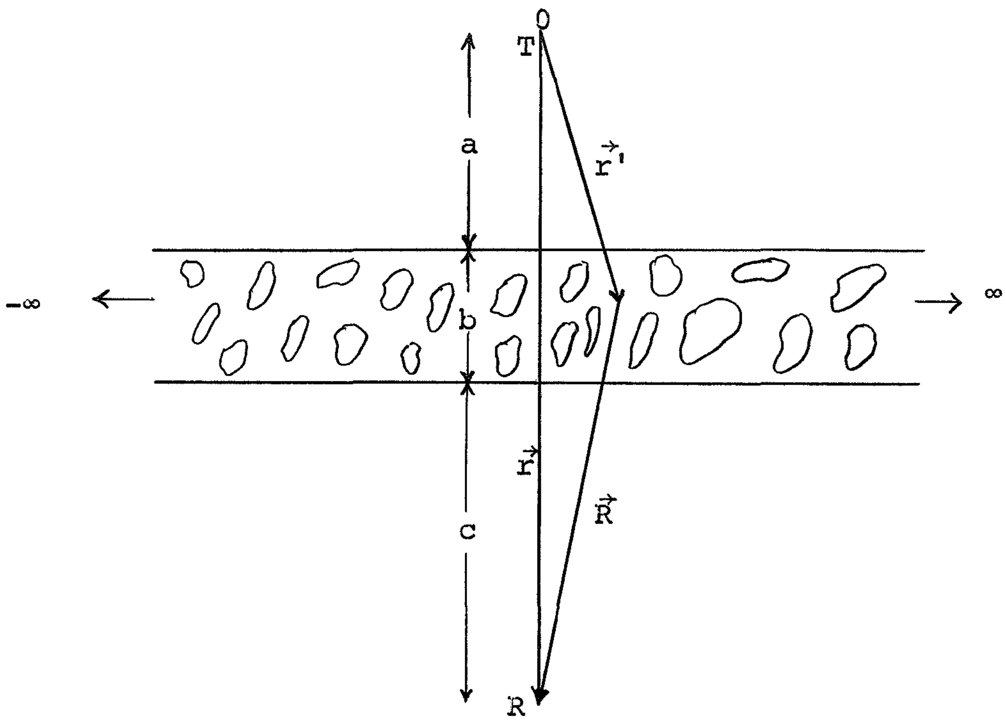


Figure 4.2.1 The geometry of the radio star problem

irregularities is bounded by the coordinates $z = a$ and $z = a + b$.

The scattered electric field at the receiver is given by equation 2.2.6 as

$$\vec{E}(\vec{r}) = -k_o^2 \int \Delta X(\vec{r}') \; \overline{\overline{\Gamma}} (\vec{r}|\vec{r}') \cdot \overline{\overline{M}} \cdot \vec{E}_i (\vec{r}') \, d\vec{r}' \qquad 4.2.1$$

Let the incident field be a characteristic wave given by

$$\vec{E}_i (\vec{r}) = A_o \, \vec{a}_i \, \exp(-j\vec{k}_i \cdot \vec{r})/r \qquad 4.2.2$$

where A_o is the amplitude of the incident wave and $\vec{a}_i$ and $\vec{k}_i$ are the characteristic wave parameters associated with the ray direction $\hat{r}$. The asymptotic Green's function is given by equation 2.5.4 as

$$\overline{\overline{\Gamma}}\ (\vec{r}|\vec{r}\,') = \frac{1}{4\pi} \sum_s \frac{d_s\ \vec{a}_s\ \vec{a}_s^{\,*}}{k_s\ \sec\ \alpha_s \sqrt{|C_s|}}\ \frac{\exp[-j\vec{k}_s\ .(\vec{r}-\vec{r}\,')]}{|\vec{r}-\vec{r}\,'|}$$

After substituting these into equation 4.2.1

$$\vec{E}(\vec{r}) = -\frac{k_0^2 A_0}{4\pi} \sum_s \int \frac{d_s (\vec{a}_s^{\,*}\ .\ \overline{\overline{M}}\ .\ \vec{a}_i)\vec{a}_s}{k_s\ \sec\ \alpha_s \sqrt{|C_s|}}$$

$$\frac{\Delta X(\vec{r}\,')\ \exp\ -j(\vec{k}_i.\vec{r}\,'+\vec{k}_s.\vec{R})}{r'\ R}\ d\vec{r}\,' \qquad 4.2.3$$

Again this equation is a very complicated function of $\vec{r}\,'$, since the parameters associated with the characteristic waves are functions of the ray directions. Therefore some approximation will be necessary.

Most of the scattered radiation will come from the first Fresnel zone. Therefore if the angular width of the first Fresnel zone, measured at the transmitter and receiver, is small, then the wave parameters can be assumed to be constant with respect to $\vec{r}\,'$ in equation 4.2.3. For an isotropic medium the radius of the first Fresnel zone is given by $r_f = \sqrt{\lambda Z}$ where λ is the wavelength and Z is the distance to the plane containing the scatterer. Therefore the angular width of the first Fresnel zone is $\phi_f = 2\tan^{-1}\sqrt{\lambda/Z}$. If the distance Z is large in terms of wavelengths, then ϕ_f is small and the above approximation is valid. More will be said about this later.

If the above condition is met then d_s, $\vec{a}_i$, $\vec{a}_s$, $\vec{k}_i$, $\vec{k}_s$, and C_s are approximately independent of $\vec{r}\,'$. Equation 4.2.3 then becomes

$$\vec{E}(\vec{r}) = -\frac{k_o^2 A_o}{4\pi} \sum_s \frac{d_s\ (\vec{a}_s^* \cdot \bar{\bar{M}} \cdot \vec{a}_i)\vec{a}_s}{k_s \sec \alpha_s \sqrt{|C_s|}} \int \frac{\Delta X(\vec{r}\,')}{r'R}$$

$$\exp -j(\vec{k}_i' \cdot \vec{r}\,' + \vec{k}_s \cdot \vec{R})\, d\vec{r}\,' \qquad 4.2.4$$

As is usual in problems of this type, remove the large portion of the phase due to the direct path by dividing equation 4.2.4 by $E_o(\vec{r}) = A_o \exp(-j\vec{k}_i \cdot \vec{r})/r$. If this is done the total received field $E_T(\vec{r})$ can be written as

$$E_T(\vec{r}) = E_o(\vec{r})\ [(1 + F_i)\ \vec{a}_i + F_s\ \vec{a}_s] \qquad 4.2.5$$

where $F_s = \frac{E_s\ (\vec{r})}{E_o(\vec{r})}$. In writing equation 4.2.5, the scattered field is assumed to be composed of a single self-scattered field and a single cross-scattered field. The possibility of more than one saddle point that contributes to the field is not taken into account. The inclusion of this possibility is easy at this stage but its interference effect is difficult to untangle later on.

Now consider

$$F_s = A_s I_s \qquad 4.2.6a$$

where

$$A_s = -\frac{k_o^2\ d_s\ (\vec{a}_s^* \cdot \bar{\bar{M}} \cdot \vec{a}_i)}{4\pi k_s \sec \alpha_s \sqrt{|C_s|}} \qquad 4.2.6b$$

and

$$I_s = \int \frac{r}{r'R} \quad \Delta X(\vec{r}') \exp\left\{-j\left(k_i (r'-r) \cos \alpha_i + k_s R \cos \alpha_s\right)\right\} dr' \qquad 4.2.6c$$

$$\equiv I_s^r + jI_s^i$$

where α is the angle between the propagation vector $\vec{k}$ and the ray path, and I_s^r and I_s^i are the respective real and imaginary parts of I_s.

The quantities of interest here are the in phase and quadriture phase components of F. These quantities are just the log amplitude and phase departure of the wave if the background medium is isotropic. Equation 4.2.6a can be separated into an in phase component F_s^p and a quadriture phase component F_s^q. These are given by

$$F_s^p = \begin{cases} A_s \; I_s^r & A_s \text{ real} \\ jA_s \; I_s^i & A_s \text{ imaginary} \end{cases} \qquad 4.2.7a$$

$$F_s^q = \begin{cases} A_s \; I_s^i & A_s \text{ real} \\ -jA_s \; I_s^r & A_s \text{ imaginary} \end{cases} \qquad 4.2.7b$$

since A_s is either real or imaginary as d_s is real or imaginary.

The correlations among F^p and F^q can now be expressed as correlations among the integrals I_s. The correlation of F^p is

$$\langle F_s^p(\vec{r}_1) F_t^p(\vec{r}_2)\rangle = \begin{cases} A_s \; A_t \; \langle I_s^r(\vec{r}_1) \; I_t^r(\vec{r}_2)\rangle \\ -jA_s \; A_t^* \; \langle I_s^r(\vec{r}_1) \; I_t^i \; (\vec{r}_2)\rangle \\ jA_s \; A_t \; \langle I_s^i(\vec{r}_1) \; I_t^r \; (\vec{r}_2)\rangle \\ A_s \; A_t^* \; \langle I_s^i(\vec{r}_1) \; I_t^i \; (\vec{r}_2)\rangle \end{cases} \qquad 4.2.8a$$

where one of the alternatives is chosen depending upon whether, A_s and A_t are both real, A_s is real and A_t is imaginary, A_s is imaginary and A_t is real or A_s and A_t are both imaginary. $\vec{r}_1$ and $\vec{r}_2$ are the locations of the two receivers. It is implicitly assumed that $\vec{r}_1$ and $\vec{r}_2$ are not too far apart so that $A_s(\vec{r}_1)$ and $A_s(\vec{r}_2)$ can be assumed to be equal. The correlation of F^q is

$$\langle F_s^q(\vec{r}_1)F_t^q(\vec{r}_2)\rangle = \begin{cases} A_s\ A_t\ \langle I_s^i(\vec{r}_1)\ I_t^i(\vec{r}_2)\rangle \\ jA_s\ A_t^*\ \langle I_s^i\ (\vec{r}_1)\ I_t^r(\vec{r}_2)\rangle \\ -jA_s\ A_t\ \langle I_s^r\ (\vec{r}_1)\ I_t^i(\vec{r}_2)\rangle \\ A_s\ A_t^*\ \langle I_s^r\ (\vec{r}_1)\ I_t^r(\vec{r}_2)\rangle \end{cases} \qquad 4.2.8b$$

The cross correlation between F^p and F^q is

$$\langle F_s^p(\vec{r}_1)\ F_t^q(\vec{r}_2)\rangle = \begin{cases} A_s\ A_t\ \langle I_s^r(\vec{r}_1)\ I_t^i(\vec{r}_2)\rangle \\ -jA_s\ A_t^*\ \langle I_s^r(\vec{r}_1)\ I_t^r(\vec{r}_2)\rangle \\ jA_s\ A_t\ \langle I_s^i(\vec{r}_1)\ I_t^i(\vec{r}_2)\rangle \\ A_s\ A_t^*\ \langle I_s^i(\vec{r}_1)\ I_t^r(\vec{r}_2)\rangle \end{cases} \qquad 4.2.8c$$

The fact that these integrals are expressible in terms of four integrals is obvious. Define these integrals as

$$I_1' = \langle I_s^r\ (\vec{r}_1)\ I_t^r\ (\vec{r}_2)\rangle \qquad 4.2.9a$$

$$I_2' = \langle I_s^i\ (\vec{r}_1)\ I_t^i\ (\vec{r}_2)\rangle \qquad 4.2.9b$$

$$I_3' = \langle I_s^i\ (\vec{r}_1)\ I_t^r\ (\vec{r}_2)\rangle \qquad 4.2.9c$$

$$I_4' = \langle I_s^r\ (\vec{r}_1)\ I_t^i\ (\vec{r}_2)\rangle \qquad 4.2.9d$$

Using equation 4.2.6c

$$I_s^r(\vec{r}_1) = \int \frac{r_1}{r'_1 R_1} \Delta X(\vec{r}'_1) \cos(T_1) \, d\vec{r}'_1 \qquad 4.2.10a$$

$$I_s^i(\vec{r}_1) = -\int \frac{r_1}{r'_1 R_1} \Delta X(\vec{r}'_1) \sin(T_1) \, d\vec{r}'_1 \qquad 4.2.10b$$

where

$$T_1 = k_i(r'_1 - r_1) \cos \alpha_i + k_s R_1 \cos \alpha_s \qquad 4.2.10c$$

Substituting these into equation 4.2.9a, it becomes

$$I'_1 = \iint \frac{r_1 r_2}{r'_1 r'_2 R_1 R_2} \langle \Delta X(\vec{r}'_1) \Delta X(\vec{r}'_2) \rangle \cos T_1 \cos T_2 \, d\vec{r}'_1 d\vec{r}'_2$$

As in Chapter III, assume $\langle \Delta X(\vec{r}_1) \Delta X(\vec{r}_2) \rangle = \langle (\Delta X)^2 \rangle B(\vec{r}'_2 - \vec{r}'_1)$.

Thus

$$I'_1 = \frac{\langle (\Delta X)^2 \rangle}{2} \iint \frac{r_1 r_2}{r'_1 r'_2 R_1 R_2} B(\vec{r}'_2 - \vec{r}'_1) \Big[\cos (T_1 - T_2) + \cos(T_1 + T_2) \Big] d\vec{r}'_1 d\vec{r}'_2 \qquad 4.2.11a$$

Similarly

$$I'_2 = \frac{\langle (\Delta X)^2 \rangle}{2} \iint \frac{r_1 r_2}{r'_1 r'_2 R_1 R_2} B(\vec{r}'_2 - \vec{r}'_1) \Big[\cos (T_1 - T_2) - \cos(T_1 + T_2) \Big] d\vec{r}'_1 d\vec{r}'_2 \qquad 4.2.11b$$

$$I'_3 = \frac{\langle (\Delta X)^2 \rangle}{2} \iint \frac{r_1 r_2}{r'_1 r'_2 R_1 R_2} B(\vec{r}'_2 - \vec{r}'_1) \Big[\sin (T_1 - T_2) + \sin (T_1 + T_2) \Big] d\vec{r}'_1 d\vec{r}'_2 \qquad 4.2.11c$$

$$I'_4 = \frac{<(\Delta X)^2>}{2} \iint \frac{r_1 r_2}{r'_1 r'_2 R_1 R_2} B(\vec{r}'_2 - \vec{r}'_1) \left[\sin (T_1 - T_2) - \sin (T_1 + T_2)\right] d\vec{r}'_1 d\vec{r}'_2 \qquad 4.2.11d$$

These equations can be expressed in terms of two fundamental integrals

$$I_1 = \iint \frac{r_1 r_2}{r'_1 r'_2 R_1 R_2} \left[B(\vec{r}'_2 - \vec{r}'_1) \exp j (T_1 - T_2)\right] d\vec{r}'_1 d\vec{r}'_2 \qquad 4.2.12a$$

$$I_2 = \iint \frac{r_1 r_2}{r'_1 r'_2 R_1 R_2} \left[B(\vec{r}'_2 - \vec{r}'_1) \exp j (T_1 + T_2)\right] d\vec{r}'_1 d\vec{r}'_2 \qquad 4.2.12b$$

Therefore equations 4.2.11 can be written as

$$I'_1 = <(\Delta X)^2> (I_1^r + I_2^r)/2 \qquad 4.2.13a$$

$$I'_2 = <(\Delta X)^2> (I_1^r - I_2^r)/2 \qquad 4.2.13b$$

$$I'_3 = -<(\Delta X)^2> (I_1^i + I_2^i)/2 \qquad 4.2.13c$$

$$I'_4 = <(\Delta X)^2> (I_1^i - I_2^i)/2 \qquad 4.2.13d$$

The problem is therefore reduced to the solution of the two integrals I_1 and I_2 .

Now consider the distances r, r', and R involved in equations 4.2.12. In general correlations transverse to the z axis lead to equations which are very complicated and therefore they will not be included. If the scattered mode is the same as the incident mode transverse correlations are possible but the mathematics is similar to the isotropic case and so it will be discussed in section 4.3. From now on it will be assumed that the two receivers are situated on the z-axis (see Fig. 4.2.1). Correlations from these two receivers are sometimes referred to as longitudinal correlations. Therefore assume that r is given by z + Z where z>>Z. The distance r' is given by $(x'^2 + y'^2 + z'^2)^{1/2}$. The Fresnel approximation (Chernov, 1960, section 18) can be applied to this because z' is given by a plus some distance in the slab, and this distance together with x' and y' are much smaller than a by a previous assumption. Thus

$$r' = z'^2 \left[1 + (x'^2+y'^2)/z\right]^{1/2}$$

$$r' \simeq z' + (x'^2+ y'^2)/2z' \qquad 4.2.14$$

The same approximation can be used on the distance R since the same statement can be made about c as was made about a (see Fig. 4.2.1). Thus

$$R = \left[(z + Z - z')^2+ x'^2+ y'^2\right]^{1/2}$$

$$R \simeq Z + z - z' + (x'^2+y'^2)/2(z - z') \qquad 4.2.15$$

With these equations the phase T can be put into a simpler form.

$$k_i(r'-r)\cos \alpha_i + k_s R \cos \alpha_s \simeq (k_s\cos\alpha_s - k_i\cos\alpha_i)(Z+z-z')$$
$$+ \left[k_i\cos\alpha_i + (k_s\cos\alpha_s - k_i\cos \alpha_i)\ z'/z\right]\frac{x'^2+ y'^2}{2z'(z-z')/z}$$

Define

$$\delta = k_s \cos \alpha_s - k_i \cos \alpha_i \qquad 4.2.16$$

$$P = k_i \cos \alpha_i + \delta z'/z \qquad 4.2.17$$

$$Q = \delta(Z + z - z') \qquad 4.2.18$$

$$\zeta = z'(z - z')/z \qquad 4.2.19$$

With these definitions

$$T = k_i\ (r'-r) \cos \alpha_i + R \cos \alpha_s = P(x'^2+y'^2)/2\zeta + Q$$

The distances which appear in the amplitude portion of equation 4.2.6c can be approximated by the largest part of the approximations given here. Thus

$$r/r'R \simeq z/z'(z-z') = 1/\zeta \qquad 4.2.21$$

Using these approximate relations, equations 4.2.12 can be written as

$$I_1 = \iint \frac{B(\vec{r}\,'_2-\vec{r}\,'_1)}{\zeta_1\zeta_2} \exp j\ (T_1 - T_2)\ d\vec{r}\,'_1\ d\vec{r}\,'_2 \qquad 4.2.22a$$

$$I_2 = \iint \frac{B(\vec{r}\,'_2-\vec{r}\,'_1)}{\zeta_1\zeta_2} \exp j\ (T_1 + T_2)\ d\vec{r}\,'_1\ d\vec{r}\,'_2 \qquad 4.2.22b$$

where T is given by equation 4.2.20. These integrals are best evaluated in the coordinate systems called relative and center of mass coordinates. The details of this transformation are given in appendix A3. Equations 4.2.22 can be written as

$$I_1 = \iiint \frac{B(\vec{r}\,')}{\zeta_1\zeta_2} \exp j\Big\{ \frac{\zeta_2 P_1 - \zeta_1 P_2}{2\zeta_1\zeta_2} \left[(\alpha'-\varepsilon_x^-)^2 + (B'-\varepsilon_y^-)^2\right] - \frac{P_1 P_2}{2(\zeta_2 P_1 - \zeta_1 P_2)} (x'^2+y'^2) + Q_1 - Q_2 \Big\} d\alpha'\, d\beta'\, d\gamma'\, d\vec{r}\,' \qquad 4.2.23a$$

$$I_2=\iint\frac{B(\vec{r}\,')}{\zeta_1\zeta_2}\exp j\left\{\frac{\zeta_2P_1+\zeta_1P_2}{2\zeta_1\zeta_2}\left[(\alpha'-\varepsilon_x^+)^2+(\beta'-\varepsilon_y^+)^2\right]\right.$$

$$\left.+\frac{P_1P_2}{2(\zeta_2P_1+\zeta_1P_2)}(x'^2+y'^2)+Q_1+Q_2\right\}d\alpha'd\beta'd\gamma'd\vec{r}\,' \qquad 4.2.23b$$

The α' and β' integrals of I_1 and I_2 have the form

$$I=\int_{-\infty}^{\infty}\exp\left[\,jH\,(x-\varepsilon)^2\right]dx$$

$$=2\int_{0}^{\infty}\exp j\,H\xi^2\,d\xi$$

From Dwight (1961) equation 858.560

$$I=\sqrt{\pi/2H}\quad(1+j) \qquad 4.2.24$$

Using equation 4.2.24, I_1 and I_2 can be written as

$$I_1=j\,2\pi\iint\frac{B(\vec{r}\,')}{\zeta_2P_1-\zeta_1P_2}\exp\left\{j\left[-\frac{P_1P_2}{2(\zeta_2P_1-\zeta_1P_2)}(x'^2+y'^2)\right.\right.$$

$$\left.\left.+Q_1-Q_2\right]\right\}d\gamma'd\vec{r}\,' \qquad 4.2.25a$$

$$I_2=j\,2\pi\iint\frac{B(\vec{r}\,')}{\zeta_2P_1+\zeta_1P_2}\exp\left\{j\left[\frac{P_1P_2}{2(\zeta_2P_1+\zeta_1P_2)}(x'^2+y'^2)\right.\right.$$

$$\left.\left.+Q_1+Q_2\right]\right\}d\gamma'd\vec{r}\,' \qquad 4.2.25b$$

Further integration of equations 4.2.25 requires a knowledge of the correlation function for the fluctuations. In a magneto-plasma such as the ionosphere this correlation function is observed to be anisotropic, in that the correlation length along the magnetic field is different from the correlation length transverse to it. Since the magnetic field can make an arbitrary angle with the plane of the slab, this correlation function will be very complicated function of the relative coordinates, the correlation lengths and the geometry. This complicates the integration so much that a closed form solution for the z related integrals does not seem possible unless a very restrictive assumption is made. For this reason a much simpler correlation function will be used. The correlation function will be assumed to be an isotropic Gaussian correlation function with correlation length ℓ. That is

$$B(\vec{r}\,') = \exp - (x'^2+y'^2+z'^2)/\ell^2 \qquad 4.2.26$$

As required by the reasons given in Chapter III, we assume $\ell \ll b$.

With this correlation function the x' and y' integrations of equations 4.2.25 have the form

$$I = \int_{-\infty}^{\infty} \exp\left[-(1/\ell^2 + jH)x^2\right] dx$$

From Chernov (1960) the footnote on page 101,

$$I = \sqrt{\pi/(1/\ell^2+jH)}$$

Using this result in equations 4.2.25, I_1 and I_2 become

$$I_1 = j4\pi^2 \iint \frac{\exp[-z'^2/\ell^2+j(Q_1-Q_2)]}{2(\zeta_2 P_1-\zeta_1 P_2)/\ell^2+jP_1P_2}\, dz'd\gamma' \qquad 4.2.27a$$

$$I_2 = j4\pi^2 \iint \frac{\exp[-z'^2/\ell^2+j(Q_1+Q_2)]}{2(\zeta_2 P_1+\zeta_1 P_2)/\ell^2-jP_1P_2}\, dz'd\gamma' \qquad 4.2.27b$$

Now consider the case when $t \neq s$. If $t \neq s$ then $t=i$ and therefore from equation 4.2.18 $Q_2 = 0$, since $\delta_2 = 0$. Therefore the phase portion with respect to γ' of equations 4.2.27 is just $\exp(-j\delta\gamma')$. The rest of the integrals is a complicated function of γ'. This function will be slowly varying compared to the phase term above because $a \leq \gamma' \leq a+b$ where a is large compared to b. Therefore as a first approximation the γ' integration becomes

$$\int_a^{a+b} \exp(-j\delta\gamma')d\gamma' = b \exp[-j\delta(a+b/2)] \sin(\delta b/2)/(\delta b/2)$$

If the differential phase across the slab δb is large this integral is negligibly small. For the cases of interest here this will be true and therefore $t \neq s$ gives a result which is very small compared to the case $t = s$. This implies that the scattered self mode and cross mode are weakly correlated if the differential phase shift across the slab is large. Therefore in the work that follows we restrict ourselves to the case $t = s$.

P, Q, and ζ in 4.2.27 are functions of z' and γ'. The calculation of the explicit dependence on z' and γ' of the various terms in equations 4.2.27 are carried out in appendix A3. Equation A3.12 gives

$$\zeta_2 P_1 - \zeta_1 P_2 = [k_i \cos\alpha_i \; (1-2\gamma'/z) - (\gamma'^2/z^2 - z'^2/4z^2)]z'$$

In this equation $a \leq \gamma' \leq a+b$ and $-b \leq z' \leq b$. Because of the weighting factor $\exp(-z'^2/\ell^2)$, the integration with respect to z' is expected to come mainly from $|z'| \lesssim \ell << a$. Hence, $z'/2$ can be ignored when compared to γ'. Therefore the above equation can be approximated by

$$\zeta_2 P_1 - \zeta_1 P_2 \simeq [k_i \cos \alpha_i \ (1-2\gamma'/z) - \delta(\gamma'/z)^2] z' \qquad 4.2.28$$

Equation A3.13 gives

$$\zeta_2 P_1 + \zeta_1 P_2 = 2z\{k_i \cos \alpha_i \ [\gamma'/z - (\gamma'/z)^2 - (z'/2z)^2]$$

$$+\delta \ (1-\gamma/z) \ [(\gamma'/z)^2 - (z'/4z)^2]\}$$

If $z'/2$ is ignored compared to γ', this equation becomes

$$\zeta_2 P_1 + \zeta_1 P_2 \simeq 2(k_i \cos\alpha_i + \delta\gamma'/z)(1-\gamma'/z)\gamma' \qquad 4.2.29$$

Equation A3.14 gives

$$P_1 P_2 = (k_i \cos \alpha_i + \delta\gamma'/z)^2 - \delta^2 (z'/2z)^2$$

which can be approximated by

$$P_1 P_2 \simeq (k_i \cos \alpha_i + \delta\gamma'/z)^2 \qquad 4.2.30$$

Also from appendix A3 equations 10 and 11

$$Q_1 - Q_2 = \delta(z' + 2Z) \qquad 4.2.31$$

$$Q_1 + Q_2 = 2\delta(z - \gamma') \qquad 4.2.32$$

Equations 4.2.27 can then be written as

$$I_1 = j4\pi^2 \exp(j2\delta Z) \int_a^{a+b} \int_{-b}^{b} \frac{\exp(-z'^2/\ell^2 + j\delta z') \ dz' d\gamma'}{2[k_i \cos\alpha_i (1-2\gamma'/z) - \delta(\gamma'/z)^2] z'/\ell^2 + j(k_i \cos\alpha_i + \delta\gamma'/z)^2} \qquad 4.2.33a$$

$$I_2 = j4\pi^2 \exp(j2\delta z) \int_a^{a+b} \int_{-b}^{b} \frac{\exp -(z'^2/\ell^2 + j2\delta\gamma') \ dz' d\gamma'}{(k_i \cos\alpha_i + \delta\gamma'/z) 4\gamma'(1-\gamma'/z)/\ell^2 - j(k_i \cos\alpha_i + \delta\gamma'/z)^2} \qquad 4.2.33b$$

Next consider the integration over z' in equations 4.2.33. The z' integrations of these equations can be extended to $\pm\infty$ since $b \gg \ell$, that is the slab is thick in terms of correlation lengths. Therefore the z' integration of I_1 has the form

$$I = \int_{-\infty}^{\infty} \frac{\exp -(z'^2/\ell^2 - j\delta z')}{(2p/\ell)(z'/\ell) + jq^2}\, dz'$$

Complete the square on z'

$$I = (\ell/2p)\ \exp[-(\ell\delta/2)^2] \int_{-\infty}^{\infty} \frac{\exp -\ (z'/\ell - j\ell\delta/2)^2}{z'/\ell\ + j\ell q^2/2p}\ dz'$$

Now transform the coordinate to $\xi = z'/\ell - j\ell\delta/2$ to get

$$I = (\ell^2/2p)\ \exp\left[\ -(\ell\delta/2)^2\right] \int_{-\infty-j\ell\delta/2}^{\infty-j\ell\delta/2} \frac{\exp(-\xi^2)}{\xi + jv}\ d\xi$$

where

$$v = \ell(q^2/p + \delta)/2$$

In order to put this equation into the form desired substitute $-\xi$ for ξ to get

$$I = -\ (\ell^2/2p)\ \exp\left[\ -(\ell\delta/2)^2\right] \int_{-\infty+j\ell\delta/2}^{\infty+j\ell\delta/2} \frac{\exp\ (-\xi^2)}{\xi - jv}\ d\xi$$

If the pole of the integrand does not lie between $j\ell\delta/2$ and the real axis, the path of integration can be deformed to

$$\int_{-\infty+j\ell\delta/2}^{-\infty} + \int_{-\infty}^{\infty} + \int_{\infty}^{\infty+j\ell\delta/2}$$ since no singularities will be

crossed. That this is possible for all cases is shown in appendix A4. At real $\xi = \pm\infty$ the integrand is zero, so two of the integrals are zero and I becomes

$$I = -(\ell^2/2p)\ \exp[-(\ell\delta/2)^2] \int_{-\infty}^{\infty} \frac{\exp(-\xi^2)}{\xi - jv}\ d\xi$$

$$= -(\sqrt{\pi}\ell^2/2p)\ \exp[-(\ell\delta/2)^2]\ P(jv)$$

where $P(\zeta)$ is the plasma disperson function defined by Fried and Conte (1961). The plasma dispersion function is related to the error function. From equation 4.2.33a

$$v = \frac{\ell}{2}\left[\frac{(k_i\ \cos\alpha_i\ + \delta\gamma'/z)^2}{k_i\ \cos\alpha_i\ (1-2\gamma'/z) - \delta(\gamma'/z)^2} + \delta\right] \qquad 4.2.34a$$

Thus

$$I_1 = j2\pi^{5/2}\ell^2\ \exp - [(\ell\delta/2)^2 - j2\delta Z]$$

$$\int_a^{a+b} \frac{P\left\{j\ \frac{\ell}{2}\left[\frac{(k_i\cos\alpha_i\ + \delta\gamma'/z)^2}{k_i\cos\alpha_i(1-2\gamma'/z) - \delta(\gamma'/z)^2} + \delta\right]\right\}}{k_i\ \cos\alpha_i\ (1-2\gamma'/z) - \delta(\gamma'/z)^2}\ d\gamma' \qquad 4.2.34b$$

The integration over z' in I_2 is very simple since only the exponential is a function of z'. Therefore from Dwight (1961) equation 860.11

$$I_2 = j4\pi^{5/2}\ell\ \exp\ (j2\delta z) \qquad 4.2.34c$$

$$\int_a^{a+b} \frac{\exp\ (-j2\delta\gamma')}{(k_i\cos\alpha_i\ + \delta\gamma'/z)(4\gamma'/\ell^2)(1-\gamma'/z) - j(k_i\cos\alpha_i + \delta\gamma'/z)^2}\ d\gamma'$$

The remaining integrals are those over the thickness of the slab for the center of mass coordinate γ'. As is usual in this type of problem they are complicated functions of γ' and therefore a further approximation is necessary. Numerical integration of these equations is possible but that is too specialized.

If the slab is very thin compared to the distances from the transmitter a and from the receiver c then γ' is approximately constant in the magnitude portions of the integrals. This thin slab model is the usual approximation made to simplify the final integration. Therefore assume that $\gamma' \simeq \tilde{\gamma} = a + b/2$ in the amplitude terms of equations 4.2.34.

From Fried and Conte (1961) the plasma dispersion function for a purely imaginary argument can be written as

$$P(jy) = j\sqrt{\pi}\,\exp(y^2)\ \operatorname{erfc}(y)$$

where erfc (y) is the complementary error function of y and y is real.

Thus I_1 can be written as

$$I_1 = 2\pi^3 \ell^2 b\,\exp[-(\ell\delta/2)^2]\,\frac{\exp(\tilde{v}^2)\ \operatorname{erfc}(\tilde{v})}{k_i\cos\alpha_i(1-2\tilde{\gamma}/z)-\delta(\tilde{\gamma}/z)^2}\,\exp(j2\delta z) \tag{4.2.35}$$

where

$$\tilde{v} = \frac{\ell}{2}\left[\frac{(k_i\cos\alpha_i+\delta\tilde{\gamma}/z)^2}{k_i\cos\alpha_i(1-2\tilde{\gamma}/z)-\delta(\tilde{\gamma}/z)^2}+\delta\right]$$

I_2 can be written as

$$I_2 = \frac{j4\pi^{5/2}\,\ell\,\exp(j2\delta z)}{(k_i\cos\alpha_i+\delta\tilde{\gamma}/z)(4\tilde{\gamma}/\ell^2)(1-\tilde{\gamma}/z)-j(k_i\cos\alpha_i+\delta\tilde{\gamma}/z)^2}\int_a^{a+b} e^{-j2\delta\gamma'}\,d\gamma'$$

$$= \frac{j4\pi^{5/2}\ell b \exp[j2\delta(z-\tilde{\gamma})]}{(k_i\cos\alpha_i+\delta\tilde{\gamma}/z)(4\tilde{\gamma}/\ell^2)(1-\tilde{\gamma}/z)-j\ (k_i\cos\alpha_i\ +\delta\tilde{\gamma}/z)^2}\ \frac{\sin\delta b}{\delta b}$$

Now consider $\sin(\delta b)/\delta b$. δb is the differential phase between the incident mode and the other mode of the plasma across the slab. This phase shift was assumed to be large when discussing the correlation between two different scattered modes. Therefore if $s\neq i$, I is very small. An examination of equation 4.2.35 shows that unless the differential phase shift in a correlation length $\delta\ell$ is large , I has an appreciable value for the case $s\neq i$. Thus

$$I_2 = \frac{j4\pi^{5/2}\ell b}{k_i^{\,2}\cos^2\alpha_i}\ \frac{1}{D(\tilde{\gamma})-j}\ \delta_{s,i} \qquad 4.2.36$$

where $\delta_{s,i}$ is the Kronecker delta and $D(\tilde{\gamma}) = 4\tilde{\gamma}(z-\tilde{\gamma})/\ell^2 z\ k_i\ \cos\alpha_i$ is the dimensionless wave parameter used by other authors (Chernov, 1960).

Now the correlations among $F_s^{\,p}$ and $F_s^{\,q}$ can be obtained. Combining equations 4.2.8, 9, and 13 these correlations become

$$\langle F_s^p\ (\vec{r}_1)\ F_s^p\ (\vec{r}_2)\rangle = \ \langle(\Delta X)^2\rangle|A_s|^2\left[I_1^r(Z)\ +\ d_s^2 I_2^r\right]/2 \qquad 4.2.37a$$

$$\langle F_s^q\ (\vec{r}_1) F_s^q\ (\vec{r}_2)\rangle = \ \langle(\Delta X)^2\rangle|\ A_s|^2\left[I_1^r(Z)\ -\ d_s^2 I_2^r\right]/2 \qquad 4.2.37b$$

$$\langle F_s^p\ (\vec{r}_1) F_s^q\ (\vec{r}_2)\rangle = \ \langle(\Delta X)^2\rangle|\ A_s|^2\left[I_1^i(Z)\ -\ d_s^2 I_2^i\right]/2 \qquad 4.2.37c$$

$\left|A_s\right|^2$ is given by (see equation 4.2.4b)

$$\left|A_s\right|^2 = \frac{k_o^4\ (\vec{a}_s^{\,*}\ \cdot\ \overline{\overline{M}}\ \cdot\ \vec{a}_i)^2}{16\pi^2\ k_s^2\ \sec^2\alpha_s\ \left|C_s\right|}$$

If the refractive index surface is convex to the ray vector, d_s is real and if the surface is concave to the ray vector, d_s is imaginary. From equation 4.2.35

$$I_1(Z) = 2\pi^3 \ell^2 b \exp[-(\ell\delta/2)^2] \frac{\exp(\tilde{v}^2)\ \mathrm{erfc}(\tilde{v})}{k_i \cos\alpha_i (1-2\tilde{\gamma}/z) - \delta(\tilde{\gamma}/z)^2}$$

$$[\cos(2\delta Z) + j \sin(2\delta Z)] \qquad 4.2.39a$$

where

$$\tilde{v} = \frac{\ell}{2}\left[\frac{(k_i \cos{}_i + \delta\tilde{\gamma}/z)^2}{k_i \cos\alpha_i\ (1-2\tilde{\gamma}/z) - \delta(\tilde{\gamma}/z)^2} + \delta\right]$$

From equation 4.2.36

$$I_2 = -\frac{4\pi^{5/2}\ell b}{k_i^2 \cos^2\alpha_i}\ \frac{1 - jD(\tilde{\gamma})}{1 + D^2(\tilde{\gamma})}\ \delta_{s,i} \qquad 4.2.39b$$

where

$$D(\tilde{\gamma}) = 4\tilde{\gamma}(z-\tilde{\gamma})/\ell^2 z k_i \cos\alpha_i .$$

Equations 4.2.37, 38 and 39 comprise the results of this section. They will be discussed in detail in the next section.

4.3 Discussion of the Results

First consider the factor $\exp[-(\ell\delta/2)^2]$ which appears in equation 4.2.39a. It can be expressed as

$$\exp[-(\ell\delta/2)^2] = \exp[-\pi^2(n_s \cos\alpha_s - n_i \cos\alpha_i)^2 \ell^2/\lambda^2] \qquad 4.3.1$$

where λ is the free space wave length. If $(n_s \cos\alpha_s - n_i \cos\alpha_i)$ is very small, $(\vec{a}_s^{\,*} \cdot M \cdot \vec{a}_i)$ will also be small (see the discussion in connection with figure 3.4.3), and therefore

$(n_s \cos\alpha_s - (n_i \cos\alpha_i)$ will have some appreciable value if cross-mode scattering is significant. If this is the case, equation 4.3.1 shows that cross-mode scattering is highly dependent upon the differential phase shift in one correlation length $\delta\ell$.

If the differential phase shift in a correlation length is large, equation 4.2.39a can be simplified. The above discussion shows that for this case I_1 is negligibly small unless s = i, and therefore there is no cross-mode scattering. A further consequence of this is that since I_1 depends upon δZ, the fields are correlated in the z direction if δ is zero and therefore equations 4.2.37 reduce to equations for the mean square values. If $\delta = 0$

$$I_1 = \frac{4\pi^{5/2}\ell b}{k_i^{\,2}\cos^2\alpha_i} \sqrt{\pi}\ \tilde{v} \exp(\tilde{v}^2)\ \mathrm{erfc}\ (\tilde{v}) \qquad 4.3.2$$

where $\tilde{v} = [\pi z n_i \cos\alpha_i/(z-2\tilde{\gamma})]\ (\ell/\lambda)$. In the case being considered here $\tilde{v}>>1$ and from Abramowitz and Stegun (1964) equation 7.1.23

$$\sqrt{\pi}\ \tilde{v} \exp\ (\tilde{v}^2)\ \mathrm{erfc}\ (\tilde{v}) \sim 1 \qquad \tilde{v} \to \infty$$

Therefore for $\ell/\lambda>>1$

$$I_1 = 4\pi^{5/2}\ell b/k_i^2 \cos^2\alpha_i$$

If the refractive index surfaces are ellipses, d_i is real. Combining equations 4.2.37, 38, 39b and 4.3.3

$$<(F_i^p)^2> = \frac{<(\Delta X)^2>\sqrt{\pi}\ \ k_o^2\ \ \ell b}{8n_i^4\ |C_i|}\ (\vec{a}_i^*.\overline{\overline{M}}.\vec{a}_i)^2\left[1 - \frac{1}{1+D^2(\tilde{\gamma})}\right] \quad 4.3.4a$$

$$<(F_i^q)^2> = \frac{<(\Delta X)^2>\sqrt{\pi}\ \ k_o^2\ \ \ell b}{8n_i^4\ |C_i|}\ (\vec{a}_i^*.\overline{\overline{M}}.\vec{a}_i)^2\left[1 + \frac{1}{1+D^2(\tilde{\gamma})}\right] \quad 4.3.4b$$

$$(F_i^pF_i^q)>=-\frac{<(\Delta X)^2>\sqrt{\pi}\ \ k_o^2\ \ \ell b}{8n_i\ |C_i|}\ \ (\vec{a}_i^*.\overline{\overline{M}}.\vec{a}_i)^2\ \ \frac{D(\tilde{\gamma})}{1 + D^2(\tilde{\gamma})} \quad 4.3.4c$$

Equations 4.3.4 a and b. compare favorably with the equations for the log amplitude $<S^2>$ and the phase departure $<Q^2>$ of Yeh and Liu (1967) and reduce to the same values for an isotropic medium.

Two special cases can be considered here. If D>>1 equations 4.3.4 reduce to

$$<(F_i^p)^2> = <(F_i^q)^2>= \frac{<(\Delta X)^2>\sqrt{\pi}k_o^2\ \ \ell b}{8n_i^4\ \ |C_i|}\ \ (\vec{a}_i^*.\overline{\overline{M}}.\vec{a}_i)^2 \quad 4.3.5a$$

$$<F_i^p\ F_i^q>= -\frac{<(\Delta X)^2>\sqrt{\pi}\ k_o^2\ell b}{8n_i^4\ \ |C_i|}\ \ (\vec{a}_i^*.\overline{\overline{M}}.\vec{a}_i)^2\ \ \frac{1}{D(\tilde{\gamma})} \quad 4.3.5b$$

Here in phase and quadriture phase components have the same mean square value. If D<<1 equations 4.3.4 become

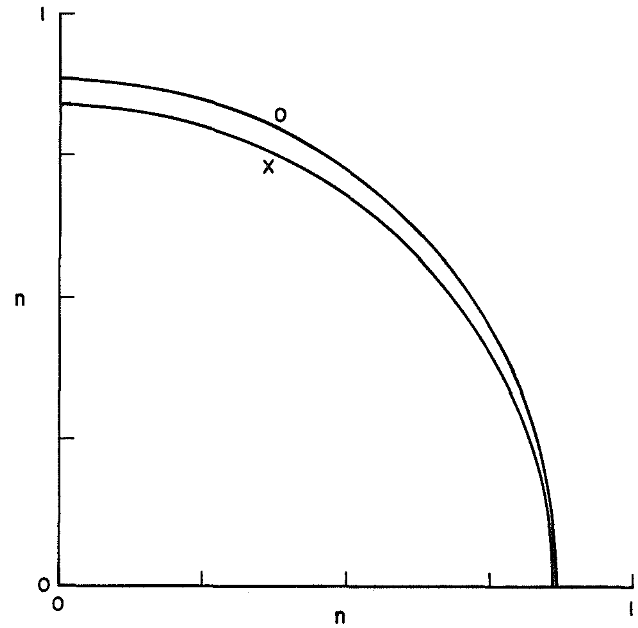
$$<(F_i^p)^2> = \frac{<(\Delta X)^2>\sqrt{\pi}\ k_o^2\ell b}{8n_i^4\ \ |C_i|}\ \ (\vec{a}_i^*.\overline{\overline{M}}.\vec{a}_i)^2\ D^2(\tilde{\gamma}) \quad 4.3.6a$$

$$<(F_i^q)^2> = \frac{<(\Delta X)^2>\sqrt{\pi}\ k_o^2\ell b}{4n_i^4\ |C_i|}\ \ (\vec{a}_i^*.\overline{\overline{M}}.\vec{a}_i)^2 \quad 4.3.6b$$

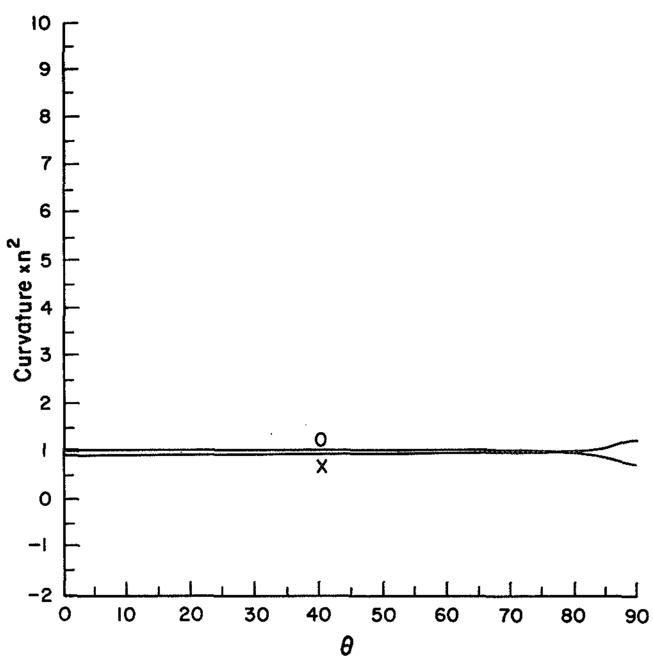
$$<F_i^p\, F_i^q> \;=\; -\,\frac{<(\Delta X)^2>\sqrt{\pi}\;k_o^2\ell b}{8n_i^4\;|c_i|}\;\;(\vec{a}_i^*.\overline{\overline{M}}.\vec{a}_i)^2\;D(\tilde{\gamma}) \qquad 4.3.6c$$

In this case the correlation of the out of phase fields is much greater than the correlation of the in phase fields.

In order to see the variation of equations 4.2.37 as a function of β, the angle the magnetic field makes with the unscattered ray path, consider the plasma shown in figure 4.3.1. In this case $X = 0.25$ and $Y = 0.15$ were chosen to correspond to the earth's ionosphere in the F region at approximately 10 Mhz. Choose the geometry such that $a = 450$ km, $b = 50$ km and $c = 500$ km. The mean square fluctuation of X is given by $[<(\Delta X)^2>/X^2]^{1/2} = 0.01$. In order to have cross mode scattering the correlation length must be approximately equal to the freespace wavelength, therefore chose $\ell = 30$ m. If ℓ is very large the previous discussion shows that the amplitude of the self mode terms will be increased but there will be no cross-mode scattering. Using the above values $D \simeq 6000$ which is the first case considered above, therefore $<F_s^{p\,2}> = <F_s^{q\,2}>$. Figure 4.3.2 shows these mean square values. The self-mode terms are shown in sections a and d and are similar in shape to the curves for small X and Y discussed in Chapter 3. The ordinary mode increases as the magnetic field becomes perpendicular to the ray path, while the extraordinary mode does just the opposite. The cross-mode terms are shown in sections b and c. They have a narrow range peaking near 80 degrees and an amplitude which is



(a) refractive index n



(b) normalized curvature

Figure 4.3.1 The refractive index and Gaussian curvature for X = 0.25 and Y = 0.15.

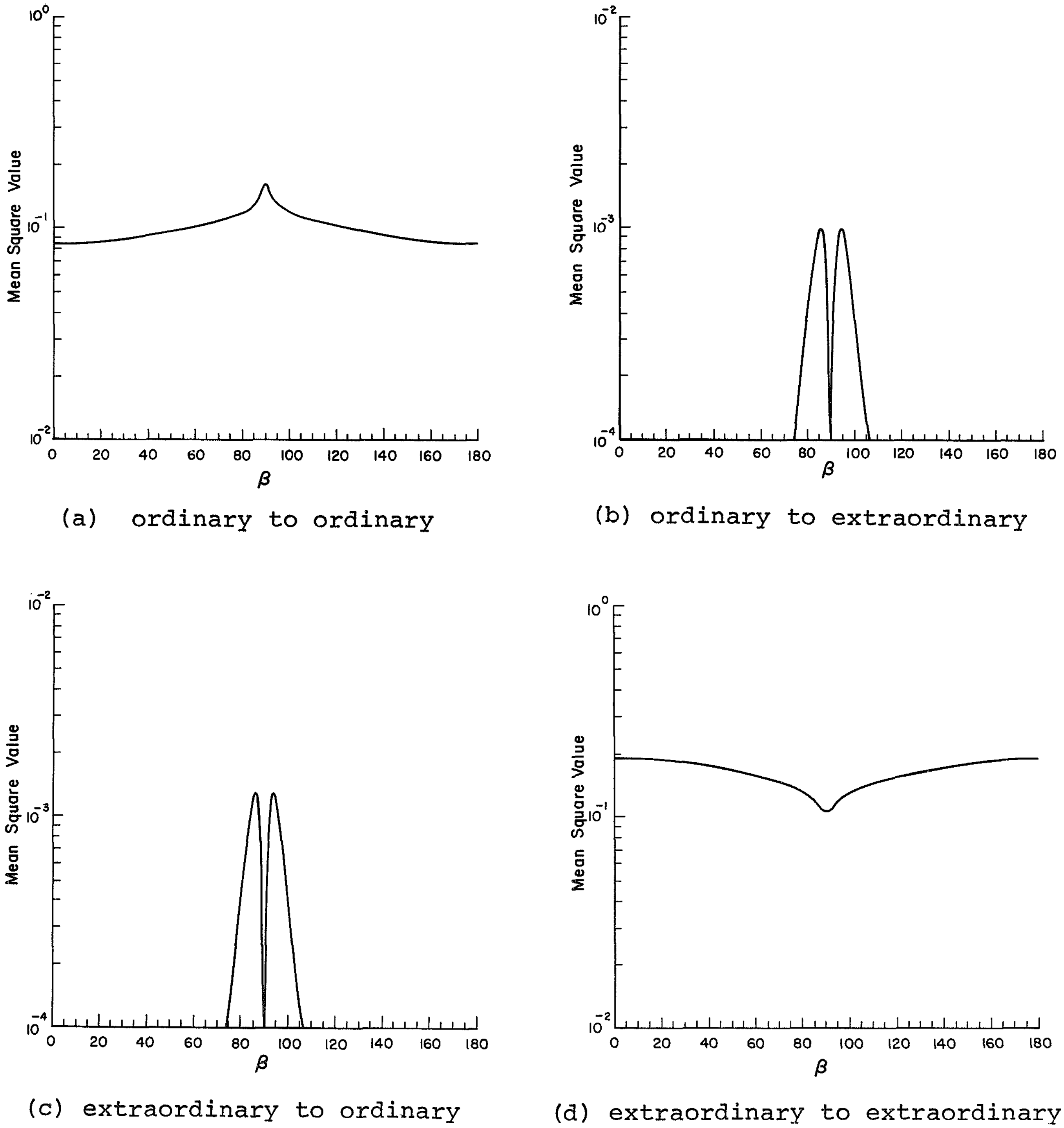


Figure 4.3.2 The mean square values of F for X = 0.25 and Y = 0.15 with $[<(\Delta X)^2>/X^2]^{1/2} = 0.01$, a = 450 km, b = 50 km and C = 500 km for a frequency of 10 Mhz.

about two orders of magnitude less than the self-mode term.

$<F_i^p\ F_s^q>$ has a value only for the self-mode terms (see equations 4.2.37 and 39b) in this case because I_1 is real for $Z = 0$. The shape of the curves are the same as the self-mode curves of figure 4.3.2 (sections a and d). The amplitudes are about four orders of magnitude less than $<(F_s^p)^2>$ and therefore much less than the cross-mode terms for most of β. They are also negatively correlated.

Equation 4.2.39a shows that the correlation distance of the amplitude is determined by $\delta = k_s\cos\alpha_s - k_i\cos\alpha_i$. This shows that δ could be determined by measuring the vertical correlation function of the cross-mode scattered fields. Thus since this term could be measured experimentally, it could be used to obtain information about the plasma in which the scattering occurs. If this is used the requirement that $\ell/\lambda \simeq 1$ must be met or there is no appreciable cross-mode scattering.

When the scattered wave is the same as the incident wave the solution is essentially the same as that for an isotropic medium. Therefore for this case transverse correlation is possible. Since the solution is essentially isotropic the transverse correlation will be the same as that determined by Chernov (1960) or Yeh (1962) multiplied by a factor which accounts for the conditions considered here. Comparing equations 4.3.4 with equations 26 and 27 of Yeh and Liu (1967) this factor is $(\bar{a}_i^*.\bar{\bar{M}}.\bar{a}_i)^2/n_i^2\ |C_i|$. The different parts of this factor are discussed in Chapter III.

4.4 The Approximations for the Slab Case

In section 3.6, the variation of the parameters associated with a characteristic wave were required to be small as the ray directions ranged over the scattering volume V. In the slab case the variation over the scattering volume is replaced by the variation of these parameters over the first Fresnel zone.

The first Fresnel zone is given by the requirement that the change in phase when going from the center to the edge of the first Fresnel zone is π radians. Thus

$$k(\vec{r})r\cos\alpha(\vec{r}) - k(\vec{r}_o)\, r_o \cos\alpha(\vec{r}_o) = \pi \qquad 4.4.1$$

The radius of the Fresnel zone r_f is perpendicular to $\bar{r}_o$ and small; therefore r can be approximated by

$$r \simeq r_o + r_f^2/2r_o \qquad 4.4.2$$

Since the variation if $\vec{k}$ over the first Fresnel zone is small, expand $k(\vec{r})\cos\alpha\ (\vec{r})$ as a Taylor's series to the first order about $\vec{r}_o$. Thus

$$k(\vec{r})\cos\alpha(\vec{r}) \simeq k(\vec{r}_o)\cos\alpha\ (\vec{r}_o) + (\vec{r}_f\cdot\nabla)[k(\vec{r})\cos\alpha(\vec{r})]_{r=r_o} \qquad 4.4.3$$

Thus equation 4.4.1 can be approximated by

$$k(\vec{r}_o)\cos\alpha(\vec{r}_o)(r_f^2/2r_o) + (\vec{r}_f\cdot\nabla)[k(\vec{r})\cos\alpha\ (\vec{r})]_{r=r_o}$$

$$\cdot(r_o + r_f^2/2r_o) = \pi \qquad 4.4.4$$

Now consider the factor $(\vec{r}_f\cdot\nabla)[k(\vec{r})\cos\alpha\ (\vec{r})]_{r=r_o}$. Both $k(\vec{r})$ and $\cos\alpha(\vec{r})$ are functions of the ray angle ρ and therefore

$$(\vec{r}_f\cdot\nabla)[k(\vec{r})\cos\alpha(\vec{r})] = (\vec{r}_f\cdot\nabla\,\rho(\vec{r}))\,\frac{d}{d\rho}\,[k(\rho)\cos\alpha(\rho)] \qquad 4.4.5$$

Using equation 2.7.3

$$\frac{d}{d\rho}\,[k(\rho)\ \cos\ \alpha(\rho)] = -k(\vec{r})\ \sin\ \alpha(\vec{r}) \qquad 4.4.6$$

ρ is given by the equation $\cos\ \rho = \hat{B}_0 \cdot \hat{r}$. From this equation

$$\nabla\rho = -(\hat{B}_0 - \cos\rho\ \hat{r})/r\ \sin\ \rho \qquad 4.4.7$$

Therefore since $\vec{r}_f \cdot \vec{r}_0 = 0$

$$(\vec{r}_f \cdot \nabla)[k(\vec{r})\ \cos\ \alpha(\vec{r})]_{r=r_0} = \frac{k(\vec{r}_0)\ \sin\ \alpha(\vec{r}_0)}{r_0\ \sin\ \rho(\vec{r}_0)}\ \hat{B}_0 \cdot \vec{r}_f$$

Because $\vec{r}_f$ is perpendicular to $\vec{r}_0$, $B_0 \cdot \vec{r}_f = \sin\ \rho(\vec{r}_0)\ r_f$.

Thus

$$(r_f \cdot \nabla)[k(\vec{r})\ \cos\ \alpha(\vec{r})]_{r=r_0} = k(\vec{r}_0)\ \sin\ \alpha(\vec{r}_0)\ r/r_0 \qquad 4.4.8$$

Using equation 4.4.8, equation 4.4.4 becomes

$$k(\vec{r}_0)\ \cos[\alpha(\vec{r}_0)]r^2/2r_0 + k(\vec{r}_0)\ \sin\ [\alpha(\vec{r}_0)]r_f = \pi \qquad 4.4.9$$

where the term $k\ \sin(\alpha)r_f^{\,3}/2r_0^2$ has been ignored in comparison with $k\ \sin(\alpha)\ r_f$. This equation can be reduced to

$$r_f^2 + 2\ \tan[\alpha(\vec{r}_0)]\ r_0 r_f - \lambda\ r_0/\ n(\vec{r}_0)\cos\ \alpha(\vec{r}_0) = 0$$

which has the solution

$$r_f = r_0\ \tan\ \alpha(\vec{r}_0) + \sqrt{\lambda r_0/n(\vec{r}_0)\cos\alpha(\vec{r}_0) + r_0^2\ \tan^2\alpha(\vec{r}_0)} \qquad 4.4.10$$

If the medium is isotropic $\alpha = 0$ and so $r_f = \sqrt{\lambda r_0/n}$ which is the radius of the first Fresnel zone for a wavelength in the medium of λ/n.

From section 3.6 equation 3.6.2 the change in ray direction $\Delta\rho$ represented by the first Fresnel zone is therefore

$$\tan(\Delta\rho) = \tan\ \alpha + \sqrt{\lambda/r_0 n(\vec{r}_0)\cos\ \alpha(\vec{r}_0) + \tan^2\alpha(\vec{r}_0)} \qquad 4.4.11$$

For small values of X and Y (especially Y) tan α is small but because of the assumption of far fields λ/r_0 is also a small number, and therefore the tan α term is significant. For larger values of α, tan α will dominate and $\Delta\rho \simeq 2\alpha$, independent of r_0.

To illustrate the above points, the plasma considered in the previous section (X = 0.25, Y = 0.15, frequency = 10 M hz) will be considered. At $\rho=0$ and $\pi/2$, $\alpha=0$ and therefore $\tan[\Delta\rho(0)] = \sqrt{\gamma/r_0 n(o)}$, which gives the value of $\Delta\rho$ = 0.34° in this case. This is the minimum angular width of the first Fresnel zone. At $\rho = \pi/4$, α = 1° which gives a value of $\Delta\rho$ = 2.11° from equation 4.4.11. This shows that there is almost an order of magnitude change in $\Delta\rho$ due to the anisotropy of the medium.

Once $\Delta\rho$ has been determined from equation 4.4.11 the rest of section 3.6 can be used to determine the degree of error introduced by using the technique described here.

The approximation introduced here is a lot more restrictive than those in Chapter 3. The reason for this is that as X and Y are increased, the size of the first Fresnel zone is dominated by α the angle between the group velocity and phase velocity. The range of α generally increases with X and Y but so does the variation of the parameters. In other words as X and Y increase, both the angular size of the first Fresnel zone and the change of characteristic parameters per unit angle increase simultaneously which is the worst condition to have occur. This shows that there should be limiting values of X and Y except for transverse and longitudinal propagation for which $\alpha = 0$ if the mode propagates.

CHAPTER V

CONCLUSIONS

5.1 Discussion

In general there are two principle conclusions that can be drawn from this work. The first deals with the appearance in the scattered field of modes different from the incident mode. Two factors determine the degree of cross mode scattering. The first is the product of the differential refractive index ($n_s \cos\alpha_s - n_i \cos\alpha_i$) with the ratio of the correlation length to the free space wavelength ℓ/λ_o. Since a wave cannot reveal structures much smaller than a wavelength (Ratcliffe, 1956, section 2.4), ℓ/λ_o will be equal to or greater than one. In general the differential refractive index will be significant since if it is small, $\vec{a}_s \cdot \overline{\overline{M}} \cdot \vec{a}_i$ will be very small and there will be no cross mode scattering. Therefore cross mode scattering is caused by irregularities whose correlation length is of the order of a wavelength. If $n_s \cos\alpha_s$ is greater than $n_i \cos\alpha_i$ there is one way in which any irregularity can cause cross mode scattering. As the wave approaches the reflection point of the s mode n_s will go to zero. Therefore near this point the differential refractive index will be zero and this factor will be a maximum for any ℓ/λ_o. The other factor is $\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i$. This factor is real and its magnitude depends upon the degree to which $\vec{a}$ differs from a circular polarization. Therefore this factor is small unless the medium is significantly anisotropic. For an isotropic medium $\vec{a}_s \cdot \overline{\overline{M}} \cdot \vec{a}_i$ is zero.

The other conclusion that can be drawn is that due to the factor $\vec{a}_s^* \cdot \overline{\overline{M}} \cdot \vec{a}_i$ the variations due to different geometries can be quite significant. The degree of variation depends upon the degree of anisotropy, especially when determined by the parameter Y. The variation is generally much less for the self mode terms than for the cross mode terms.

From the standpoint of the restrictiveness of the approximations required, the bistatic geometry is more versatile than the radio star geometry. This is true because the scattering volume is controlled by the experiment in the bistatic case while the effective scattering volume is controlled by the background medium in the radio star case. Also in the radio star problem the effective scattering volume as well as the rate of change of the parameters increases as the medium becomes highly anisotropic. This is a very restrictive situation to have occur.

5.2 Future Work

Some areas for future work have already been suggested but they will be repeated here. A better asymptotic Green's function should be developed to account for vanishing Gaussian curvature at points of inflection. This would be very useful because the beams which result from points of inflection decrease slower with distance than regular fields do. An attempt could be made to use a warm plasma theory to get better results near asymptotes. Because of the flat nature of the refractive index in these regions beams will be generated and therefore this extension would be useful. The solution might be formulated in terms of linearly polarized waves to obtain some more information about Faraday rotation. An attempt might also be made to apply some of the techniques of multiple scattering. Finally the phase approximation might be improved by using a Taylor's series expansion of the wave number. The results of this look very complicated and so this may not be a useful extension.

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APPENDIX A1

THE CHARACTERISTIC FIELD $\vec{a}$ IN A GENERAL COORDINATE SYSTEM

The basic expressions for $\vec{a}$ are given in Chapter II, section 6 and they are repeated here. In the coordinate system shown in Figure A1.1

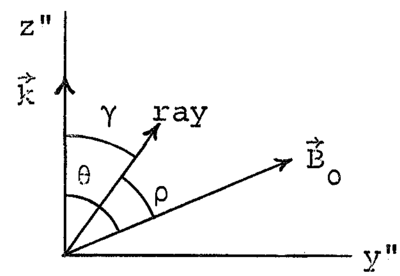

Figure A1.1 Special coordinate system for $\vec{a}$.

$\vec{a}$ is given by

$$\vec{a} = \frac{1}{[1+|R|^2]^{1/2}} [R, 1, RQ] \qquad \text{A1.1}$$

where R and Q are given by equations 2.6.4b and 2.6.4c. This vector is to be transformed into a coordinate system shown in Figure A1.2.

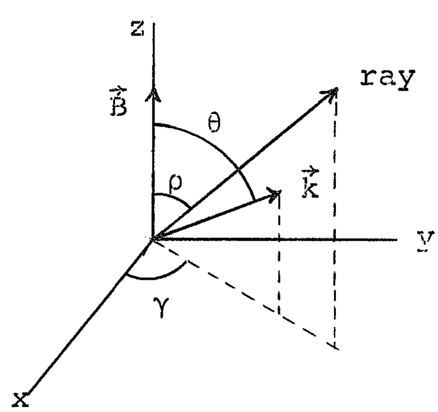

Figure A1.2 General coordinate system for $\vec{a}$.

This is just a rotation about the x"-axis through an angle θ and then a rotation about the z'-axis through an angle γ + 90. The form of the first rotation can be determined from Figure A1.3.

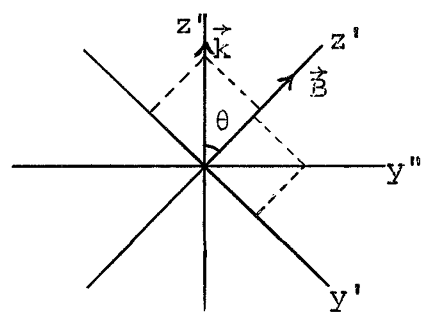

Figure A1.3 The rotation about x.

It is

$$\vec{r}\,' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \vec{r}\,'' \qquad \text{A1.2}$$

The form of the rotation about the new z axis can be determined from Figure A1.4. It is

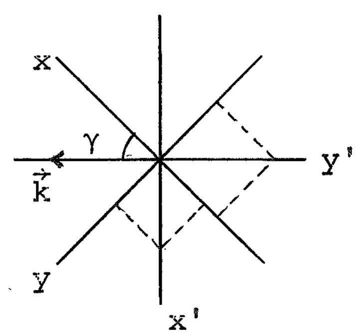

Figure A1.4 The rotation about z.

$$\vec{r} = \begin{bmatrix} -\sin\gamma & -\cos\gamma & 0 \\ \cos\gamma & -\sin\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{r}' \qquad \text{A1.3}$$

Therefore from equations A1.2 and A1.3 the total rotation can be written as

$$\vec{r} = \begin{bmatrix} -\sin\gamma & -\cos\gamma & 0 \\ \cos\gamma & -\sin\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \vec{r}''$$

$$= \begin{bmatrix} -\sin\gamma & -\cos\theta\cos\gamma & \sin\theta\cos\gamma \\ \cos\gamma & -\cos\theta\sin\gamma & \sin\theta\sin\gamma \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \vec{r}''$$

Thus the characteristic field $\vec{a}$ in the coordinate of Figure A1.2 becomes

$$\vec{a} = \frac{1}{[1+|R|^2]^{1/2}} \begin{bmatrix} -\sin\gamma & -\cos\theta\cos\gamma & \sin\theta\cos\gamma \\ \cos\gamma & -\cos\theta\sin\gamma & \sin\theta\sin\gamma \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} R \\ 1 \\ RQ \end{bmatrix}$$

$$= \frac{1}{[1+|R|^2]^{1/2}} \begin{bmatrix} R(Q\sin\theta\cos\gamma - \sin\gamma) - \cos\theta\cos\gamma \\ R(Q\sin\theta\sin\gamma + \cos\gamma) - \cos\theta\sin\gamma \\ RQ\cos\theta + \sin\theta \end{bmatrix} \qquad \text{A1.4}$$

APPENDIX A2

POLAR PLOTS OF G_s^i FOR A PLASMA WITH X = 0.2 AND Y = 0.4

Figures A2.1 - A2.4 are polar plots of the geometric factor G_s^i for a plasma with X = 0.2 and Y = 0.4. The geometric factor is plotted as a function of ϕ_s the angle the scattered ray makes with the magnetic field in the plane defined by the magnetic field and the incident ray. The two self mode terms are plotted for $0° \leq \phi_s \leq 180°$. The two cross mode terms are plotted for $180° \leq \phi_s \leq 360°$ and are ten times larger than they actually are. In Figure A2.1 the incident ray is along the magnetic field. In Figure A2.2 the incident ray makes an angle of 30° with the magnetic field. In Figure A2.3 the incident ray is inclined to $\vec{B}_0$ with an angle of 60°. In Figure A2.4 the incident ray and the magnetic field are perpendicular.

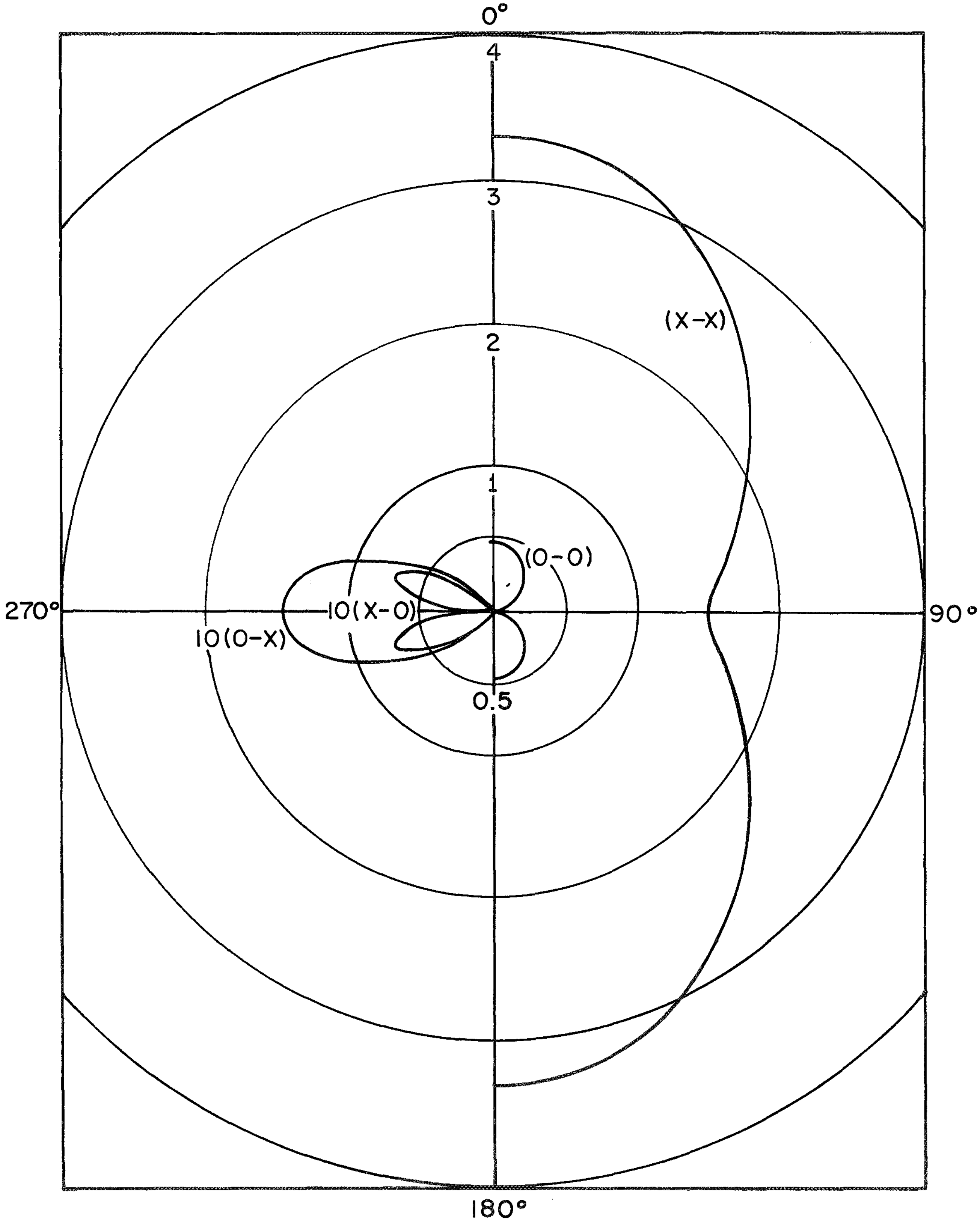

Figure A2.1 The geometric factor versus the scattered ray angle for an incident ray angle of 0°.

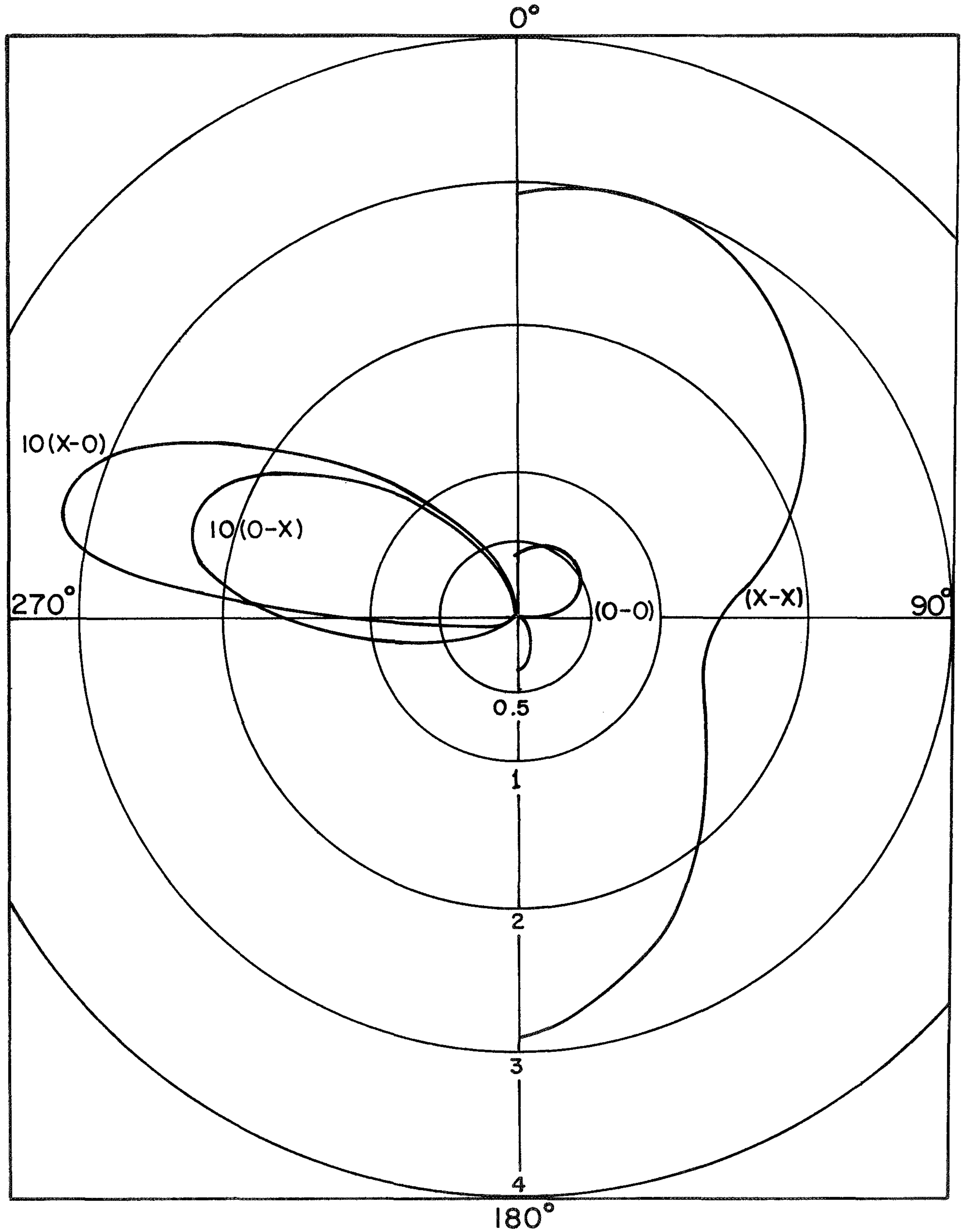

Figure A2.2 The geometric factor versus the scattered ray angle for an incident ray angle of 30°.

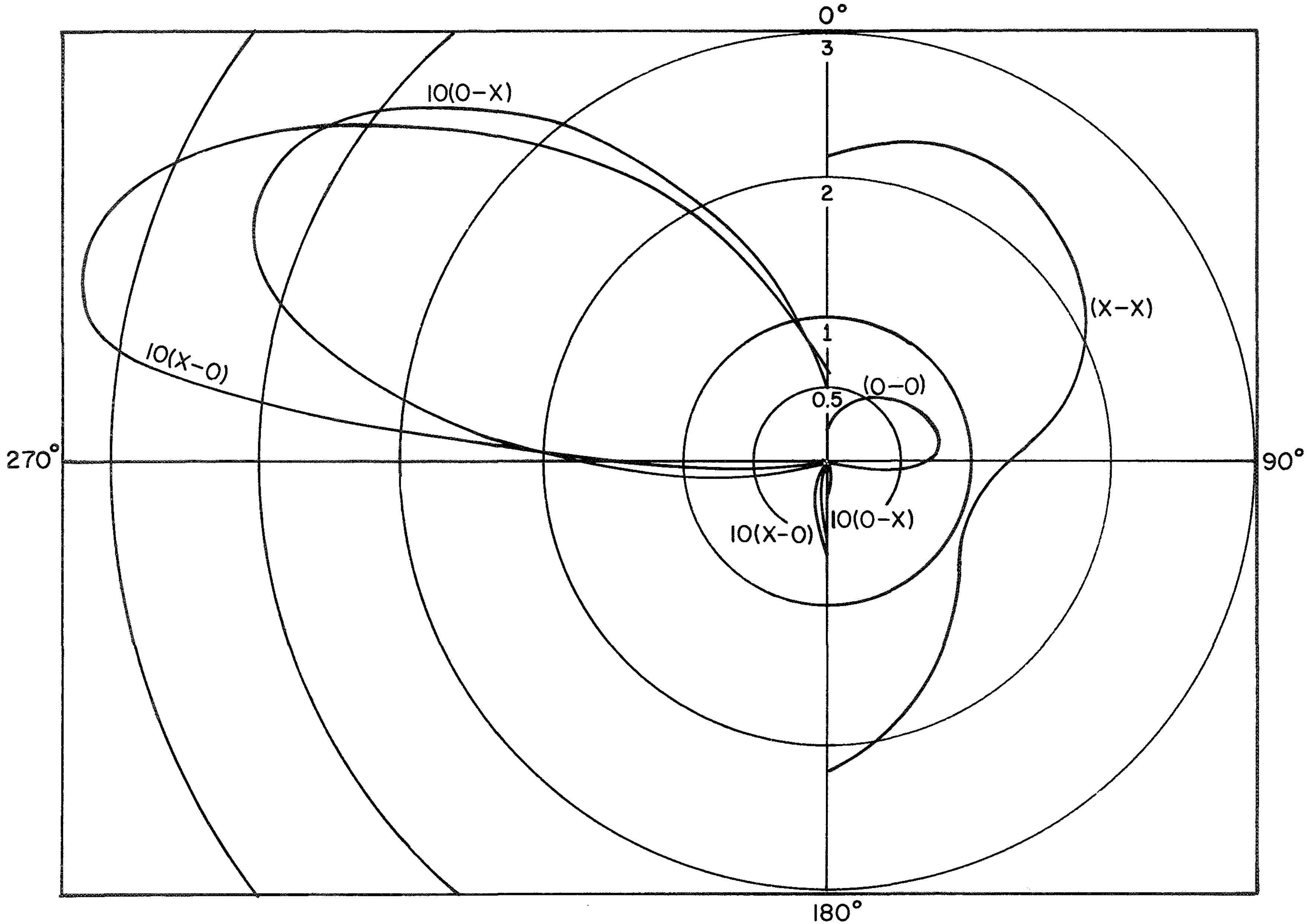

Figure A2.3 The geometric factor versus the scattered ray angle for an incident ray angle of 60°.

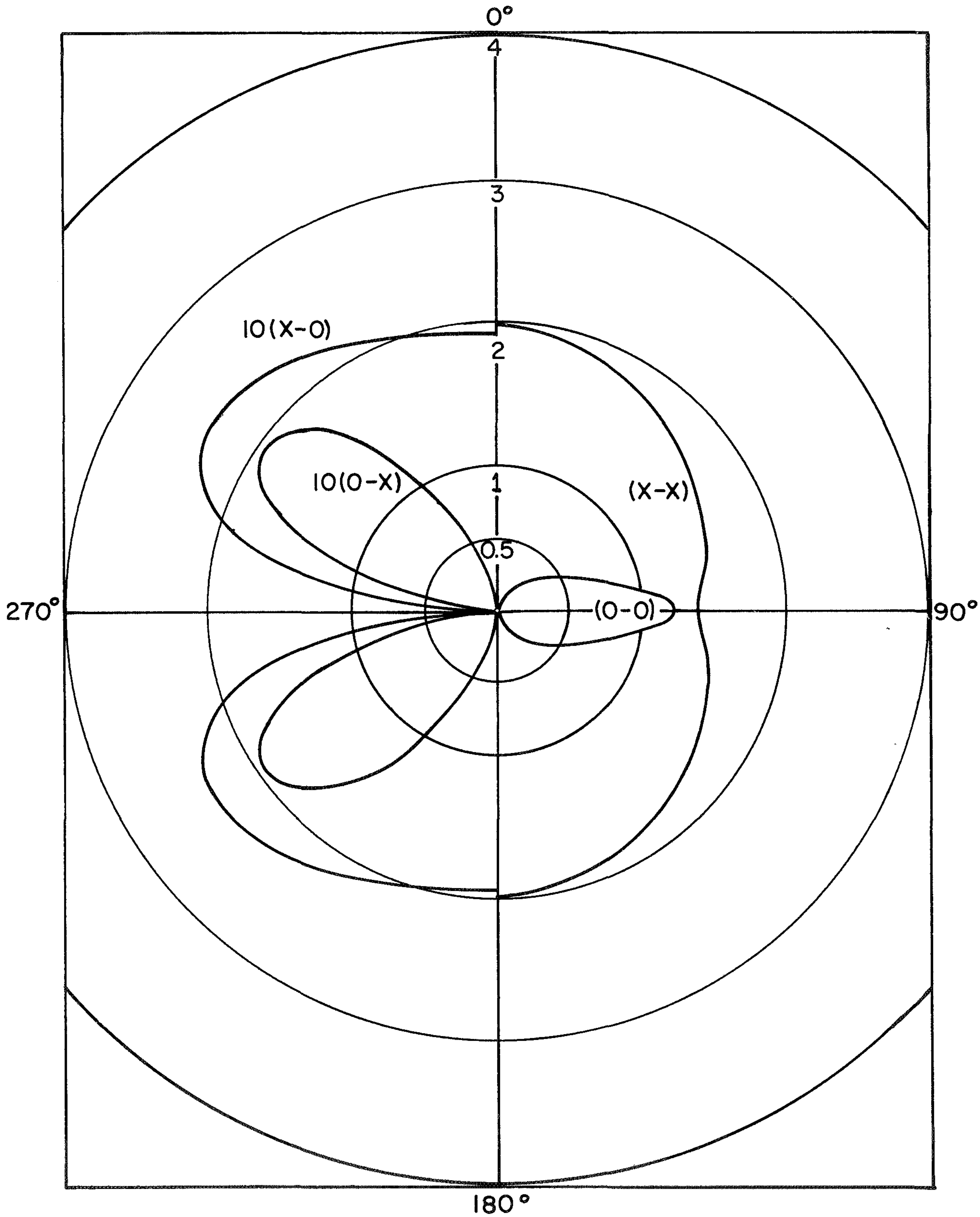

Figure A2.4 The geometric factor versus the scattered ray angle for an incident ray angle of 90°.

APPENDIX A3

THE TRANSFORMATION OF T_1-T_2 and T_1+T_2 INTO RELATIVE AND CENTER OF MASS COORDINATES

The relative coordinates are defined by

$$x' = x_2'-x_1' \qquad y' = y_2'-y_1' \qquad z' = z_2'-z_1' \tag{A3.1}$$

The center of mass coordinates are

$$2\alpha' = x_2'+x_1' \qquad 2\beta' = y_2'+y_1' \qquad 2\gamma' = z_2'+z_1' \tag{A3.2}$$

In these coordinates $\vec{r}_1'$ is given by

$$x_1' = \alpha'-x'/2 \quad y_1' = \beta'-y'/2 \qquad z_1' = \gamma'-z'/2 \tag{A3.3}$$

and $\vec{r}_2'$ by

$$x_2' = \alpha'+x'/2 \quad y_2' = \beta'+y'/2 \qquad z_2' = \gamma'+z'/2 \tag{A3.4}$$

Now

$$T_1-T_2 = (P_1/2\zeta_1)(x_1'^2+y_1'^2) + Q_1 - (P_2/2\zeta_2)(x_2'^2+y_2'^2) - Q_2 \tag{A3.5}$$

Consider the terms

$$(P_1/2\zeta_1)x_1'^2-(P_2/2\zeta_2)x_2'^2 = (1/2)[(P_1/\zeta_1)(\alpha'-x'/2)^2 - (P_2/\zeta_2)(\alpha'+x'/2)^2]$$

$$= \frac{\zeta_2 P_1-\zeta_1 P_2}{2\zeta_1\zeta_2}\left(\alpha'^2 - \frac{\zeta_2 P_1+\zeta_1 P_2}{\zeta_2 P_1-\zeta_1 P_2}x'\alpha' + \frac{x'^2}{4}\right)$$

Complete the square on α' and collect terms.

$$(P_1/2\zeta_1)x_1'^2-(P_2/2\zeta_2)x_2'^2 = \frac{\zeta_2 P_1-\zeta_1 P_2}{2\zeta_1\zeta_2}(\alpha'-\varepsilon_x^-)^2-\frac{P_1 P_2}{2(\zeta_2 P_1-\zeta_1 P_2)}x'^2 \tag{A3.6}$$

where $\varepsilon_x^- = \dfrac{\zeta_2 P_1+\zeta_1 P_2}{\zeta_2 P_1-\zeta_1 P_2}\dfrac{x'}{2}$. Let $\varepsilon_y^- = \dfrac{\zeta_2 P_1+\zeta_1 P_2}{\zeta_2 P_1-\zeta_1 P_2}\dfrac{y'}{2}$, then equation A3.5 can be reduced to

$$T_1-T_2 = \frac{\zeta_2P_1-\zeta_1P_2}{2\zeta_1\zeta_2}\left[(\alpha'-\varepsilon_x^-)^2+(\beta^1-\varepsilon_y^-)^2\right]$$

$$-\frac{P_1P_2}{2(\zeta_2P_1-\zeta_1P_2)}(x'^2+y'^2)+Q_1-Q_2 \qquad \text{A3.7}$$

Now consider T_1+T_2

$$T_1+T_2 = (P_1/2\zeta_1)(x_1'^2+y_1'^2)+Q_1+(P_2/2\zeta_2)(x_2'^2+Y_2'^2)+Q_2 \qquad \text{A3.8}$$

This is just equation A3.5 with P_2 replaced by $-P_2$ and Q_2 replaced by $-Q_2$. Therefore from equation A3.7

$$T_1+T_2 = \frac{\zeta_2P_1+\zeta_1P_2}{2\zeta_1\zeta_2}\left[(\alpha'-\varepsilon_x^+)^2+(\beta'-\varepsilon_y^+)^2\right]$$

$$+\frac{P_1P_2}{2(\zeta_2P_1+\zeta_1P_2)}(x'^2+y'^2)+Q_1+Q_2 \qquad \text{A3.9}$$

where $\varepsilon_x^+ = \dfrac{\zeta_2P_1-\zeta_1P_2}{\zeta_2P_1+\zeta_1P_2}\,\dfrac{x'}{2}$ and $\varepsilon_y^+ = \dfrac{\zeta_2P_1-\zeta_1P_2}{\zeta_2P_1+\zeta_1P_2}\,\dfrac{y'}{2}$.

Now consider the term (see equation 4.2.11)

$$Q_1-Q_2 = \delta(z-z_1'+Z-z+z_2'+Z) \qquad \text{A3.10}$$

$$= \delta(z'+2Z)$$

also

$$Q_1+Q_2 = \delta(z-z_1'+Z+z-z_2'-Z)$$

$$=2\delta(z-\gamma') \qquad \text{A3.11}$$

Consider the term (see equations 4.2.17 and 4.2.19)

$$\zeta_2P_1-\zeta_1P_2 = \frac{z_2'(z-z_2')}{z}\left(k_i\cos\alpha_i+\delta\frac{z_1'}{z}\right)-\frac{z_1'(z-z_1')}{z}\left(k_i\cos\alpha_i+\delta\frac{z_2'}{z}\right)$$

$$=\frac{k_i\cos\alpha_i}{z}\,(zz_2'-z_2'^2-zz_1'+z_1'^2) + \frac{\delta z_1'z_2'}{z^2}\,(z-z_2'-z+z_1')$$

$$= k_i z'\cos\alpha_i - 2k_i(\gamma' z'/z)\cos\alpha_i - (\delta z'/z^2)(\gamma'^2 - z'^2/4) \qquad \text{A3.12}$$

also

$$\zeta_2 P_1 + \zeta_1 P_2 = \frac{z_2'(z-z_2')}{z}\left(k_i\cos\alpha_i + \delta\frac{z_1'}{z}\right) + \frac{z_1'(z-z_1')}{z}\left(k_i\cos\alpha_i + \delta\frac{z_2'}{z}\right)$$

$$= \frac{k_i\cos\alpha_i}{z}\,(zz_2'-z_2'^2+zz_1'-z_1'^2) + \frac{\delta z_1'z_2'}{z^2}\,(z-z_2'+z-z_1')$$

$$=2k_i\gamma'\cos\alpha_i - 2k_i\frac{\gamma'^2+z'^2/4}{z}\cos\alpha_i + 2\delta\frac{z-\gamma'}{z^2}\left(\gamma'^2-\frac{z'^2}{4}\right) \qquad \text{A3.13}$$

Finally

$$P_1P_2 = [k_i\cos\alpha_i + \delta(\gamma'-z'/2)/z]\,[k_i\cos\alpha_i + \delta(\gamma'+z'/2)/z]$$

$$= (k_i\cos\alpha_i + \delta\gamma'/z)^2 - (\delta z'/2z)^2 \qquad \text{A3.14}$$

APPENDIX A4

LOCATION OF THE POLE OF THE INTEGRAND OF EQUATION 4.2.33a

The pole of the integrand is at

$$\xi = j\ell(q^2/p + \delta)/2 \qquad \text{A4.1}$$

where
$$\delta = k_s\cos\alpha_s - k_i\cos\alpha_i$$
$$p = k_i\cos\alpha_i(1-2\gamma'/z) - \delta(\gamma'/z)^2$$
$$q = k_i\cos\alpha_i + \delta\gamma'/z$$

The path of integration is along the line

$$\xi = j\,\ell\delta/2$$

Therefore $\Lambda = q^2/p + \delta$ will be compared to δ to see if it lies between δ and the origin. If it does not the new path of integration can be used.

First expand Λ to get its dependence on the different parameters.

$$\Lambda = \frac{(k_i\cos\alpha_i + \delta\gamma'/z)^2}{k_i\cos\alpha_i(1-2\gamma'/z) - \delta(\gamma'/z)^2} + \delta$$

$$= \frac{k_i^2\cos^2\alpha_i + \delta k_i\cos\alpha_i}{k_i\cos\alpha_i(1-2\gamma'/z)-\delta(\gamma'/z)^2}$$

$$= \frac{k_ik_s\cos\alpha_i\ \cos\alpha_s}{k_i\cos\alpha_i(1-\gamma'/z)^2 - k_s\cos\alpha_s(\gamma'/z)^2}$$

Define $\kappa = k_s\cos\alpha_s/k_i\cos\alpha_i$ and $\varepsilon = \gamma'/z$. Thus

$$\Lambda = \frac{\kappa/\varepsilon^2}{(1-\varepsilon)^2/\varepsilon^2-\kappa}\quad k_i\cos\alpha_i \qquad \text{A4.2}$$

δ can be written as

$$\delta = (\kappa-1)\ k_i\cos\alpha_i \qquad \text{A4.3}$$

Now if δ is positive and Λ is greater than δ or less than zero the change of path is possible. From equation A4.2, Λ is less than zero if

$$\kappa > (1-\varepsilon)^2/\varepsilon^2$$

Since δ is positive $\kappa>1$, therefore if $\varepsilon>.5$ the above inequality is always satisfied. Since q^2 is always positive, Λ will be greater than δ if p is positive. p is essentially the denominator of equation A4.2 and is positive if

$$\kappa< (1-\varepsilon)^2/\varepsilon^2$$

Therefore if δ is positive the change of path is always possible, since for δ positive

$$\kappa< (1-\varepsilon)^2/\varepsilon^2 \qquad \Lambda>\delta \tag{A4.4a}$$

$$\kappa> (1-\varepsilon)^2/\varepsilon^2 \qquad \Lambda<0 \tag{A4.4b}$$

For δ negative, Λ must be less than δ or greater than zero. Λ is greater than zero if

$$\kappa< (1-\varepsilon)^2/\varepsilon^2$$

Λ is less than δ if p is negative. p is negative if

$$\kappa> (1-\varepsilon)^2/\varepsilon^2$$

Therefore if δ is negative the change of path is always possible again, since for δ negative

$$\kappa< (1-\varepsilon)^2/\varepsilon^2 \qquad \Lambda>0 \tag{A4.5a}$$

$$\kappa> (1-\varepsilon)^2/\varepsilon^2 \qquad \Lambda<\delta \tag{A4.5b}$$

The above discussion shows that Λ never lies between δ and the origin and therefore the path of integration can always be deformed as desired.

APPENDIX A 5

COMPUTER PROGRAMS

Some of the computer programs used in developing this report are presented here.

The first program, titled Refractive Index Parameters 0-90, computes the value of ρ, the refractive index n and the Gaussian curvature C at a given value of θ for both modes. The input card(s) contains X, Y, frequency (optional) and the increment in θ. The output cards contain θ, ρ, n, and C first for the ordinary mode then for the extraordinary mode. ρ always has a value from 0° to 90° and θ is adjusted accordingly. These cards are then used to do the actual computation work.

The next program TALOK is a subroutine which uses the values from the previous program. For a given value of ρ it scans the table generated previously to determine the values of θ, n, and C which correspond to it. RHOD is the input value of ρ. THATA, N and C are the output values of θ, n, and C. They are in a 2 x 3 array. The first index refers to the mode (1 = ordinary, 2 = extraordinary), the second to the stationary points for that mode. KS gives the number of values for each mode. In this program ρ can have any value.

The last program CARFLD is a subroutine to compute the characteristic fields. It uses the results of the previous subroutine. The input values are X, Y, θ, γ, the azimuthal angle between the ray vector and $\vec{B}_o$, n, and KS above. The output is A the characteristic vectors and B the magnitude of A. A is a 2 x 3 x 3 matrix. The first index is for the mode, the second is for the stationary point and the third is the components of the vector in the coordinate system of figure A1.2.

```
C     REFRACTIVE INDEX PARAMETERS  0- 90
      IMPLICIT REAL(N)
      DIMENSION T(2),TP(2),TPP(2),NN(2),NPP(2),AL(2),ALD(2),RHO(2),
     1THAA(2)
      COMMON THATA(2,91),RHOD(2,91),N(2,91),G(2,91),MM,X,Y
      RD=57.29578
      HFPI=1.570796
5     READ(5,100,END=99) X,Y,FREQ,DM
100   FORMAT(4(F10.5))
      PUNCH 100,X,Y,FREQ,DM
      PRINT 101,X,Y,FREQ
101   FORMAT('1',5X,'X=',F10.5,5X,'Y=',F10.5,5X,'FREQ=',F7.2,'MHZ')
      PRINT 102
102   FORMAT('0',28X,'ORDINARY',42X,'EXTRAORDINARY')
      PRINT 103
103   FORMAT(' ',2('THETA',6X,'RHO',8X,'T',9X,'N',8X,'GAU CURV',6X))
      DEL=DM*0.01745329
      THA=-DEL
      MM=(90.0/DM)+1
      DO 2 J=1,MM
      THA=THA+DEL
      ST=Y*SIN(THA)
      CT=Y*COS(THA)
      XO=1.0-X
      A=ST*ST/(2.0*XO)
      SQ=SQRT(A*A+CT*CT)
      B=ST*CT/XO
      C=(A-XO)/SQ
      D=(CT*CT-ST*ST)/XO
      E=C*D+B*B*(1.0-C*C)/SQ
      T(1)=A-SQ
      T(2)=A+SQ
      TP(1)=B*(1.0-C)
      TP(2)=B*(1.0+C)
      TPP(1)=D-E
      TPP(2)=D+E
      DO 1 I=1,2
      TT=1.0-T(I)
      NN(I)=1.0- (X/TT)
      IF (NN(I)) 6,6,7
6     RHO(I)=-1.0
      RHOD(I,J)=-1.0
      THAA(I)=0.0
      THATA(I,J)=0.0
      N(I,J)=0.0
      G(I,J)=0.0
      GO TO 1
7     IF (NN(I)-100.0) 21,20,20
20    RHO(I)=-1.0
      RHOD(I,J)=-1.0
      THAA(I)=0.0
      THATA(I,J)=0.0
      N(I,J)=-1.0
      G(I,J)=-1.0
      GO TO 1
21    N(I,J)=SQRT(NN(I))
      T1=2.0+(X/(2.0*(TT-X)))
```

```
      T2=T1*TP(I)*TP(I)/TT
      T3=-X/(2.0*N(I,J)*TT*TT)
      NPP(I)=T3*(TPP(I)+T2)
      AL(I)=ATAN(X*TP(I)/(2.0*TT*(TT-X)))
      RHO(I)=AL(I)+THA
      IF(ST) 3,4,3
4     F=1.0-(NPP(I)/N(I,J))
      GO TO 10
3     F=SIN(RHO(I))*COS(AL(I))/SIN(THA)
10    G(I,J)=F*(1.0+SIN(AL(I))**2-(NPP(I)*(COS(AL(I))**2))/N(I,J))/NN(I)
      IF (RHO(I).LT.0.0) RHO(I)=4.0*HFPI+RHO(I)
      IF (RHO(I)-    HFPI) 13,13,11
11    IF (RHO(I)-2.0*HFPI) 14,14,12
12    IF (RHO(I)-3.0*HFPI) 15,15,16
13    M=1
      RDEL=0.0
      GO TO 17
14    M=-1
      RDEL=2.0*HFPI
      RHO(I)=RDEL-RHO(I)
      GO TO 17
15    M=1
      RDEL=2.0*HFPI
      RHO(I)=RHO(I)-RDEL
      GO TO 17
16    M=-1
      RDEL=4.0*HFPI
      RHO(I)=RDEL-RHO(I)
17    THAA(I)=RDEL+M*THA
      IF (THAA(I).LT.0.0) THAA(I)=4.0*HFPI+THAA(I)
      THATA(I,J)=THAA(I)*RD
      RHOD(I,J)=RHO(I)*RD
1     CONTINUE
90    PRINT 104,(THATA(I,J),RHOD(I,J),T(I),N(I,J),G(I,J),I=1,2)
104   FORMAT('0',2(2(F6.2,4X),2(F7.4,3X),F10.5,4X))
      PUNCH 105,(THAA(I),RHO(I),N(I,J),G(I,J),I=1,2)
105   FORMAT(3F7.4,F10.5,3X,3F7.4,F10.5)
2     CONTINUE
      GO TO 5
99    CALL EXIT
      END
```

```
      SUBROUTINE TALOK(RHOD,THATA,N,C,KS)
      IMPLICIT REAL(N)
      COMMON/IN/THA(2,181),RH(2,181),RI(2,181),CV(2,181),MM
      DIMENSION THATA(2,3),          N(2,3),C(2,3),O(2),KS(2),KK(2)
      HFPI=1.570796
      IF (RHOD-HFPI) 51,51,52
51    MR=1
      RDEL=0.0
      GO TO 53
52    MR=-1
      RDEL=2.0*HFPI
53    RHO=RDEL+MR*RHOD
      IF (RHO.LT.0.0) RHO=-RHO
22    DO 4 I=1,2
      KS(I)=0
      IS=0
      K=1
12    IF (RH(I,K)) 10,11,11
10    IF(K-MM)13,14,13
13    K=K+1
      GO TO 12
14    KK(I)=-1
      GO TO 4
11    IF (RHO-RH(I,K)) 1,2,3
1     O(I)=-1.0
      KK(I)=K+1
      GO TO 4
3     O(I)=1.0
      KK(I)=K+1
      GO TO 4
2     KS(I)=1
      IS=1
      O(I)=-1.0
      KK(I)=K+1
      THATA(I,IS)=THA(I,K)
      THATA(I,IS)=RDEL+MR*THATA(I,IS)
      IF (THATA(I,IS).LT.0.0) THATA(I,IS)=4.0*HFPI+THATA(I,IS)
      N(I,IS)=RI(I,K)
      C(I,IS)=CV(I,K)
4     CONTINUE
      DO 99 I=1,2
      K2=KK(I)
      IF(K2) 99,99,16
16    DO 8 J=K2,MM
      IS=KS(I)
      IF (RH(I,J)) 8,15,15
15    A=RHO-RH(I,J)
      IF (A*O(I)) 5,6,8
6     IS=IS+1
      KS(I)=KS(I)+1
      O(I)=-O(I)
      THATA(I,IS)=THA(I,J)
      N(I,IS)=RI(I,J)
      C(I,IS)=CV(I,J)
      GO TO 7
5     IS=IS+1
      KS(I)=KS(I)+1
      O(I)=-O(I)
      FAC=(RHO-RH(I,J-1))/(RH(I,J)-RH(I,J-1))
      FAC1=1.0-FAC
      THATA(I,IS)=FAC1*THA(I,J-1)+FAC*THA(I,J)
      N(I,IS)=FAC1*RI(I,J-1)+FAC*RI(I,J)
      C(I,IS)=FAC1*CV(I,J-1)+FAC*CV(I,J)
7     THATA(I,IS)=RDEL+MR*THATA(I,IS)
      IF (THATA(I,IS).LT.0.0) THATA(I,IS)=4.0*HFPI+THATA(I,IS)
      IF (IS-3) 8,99,99
8     CONTINUE
99    CONTINUE
      RETURN
      END
```

```
      SUBROUTINE CARFLD(X,Y,THATA,GAMA,N,K,A,B)
      COMPLEX A(2,3,3),R,Q
      REAL N(2,3)
      DIMENSION THATA(2,3),B(2,3),S(2),K(2)
      S(1)=-1.0
      S(2)=1.0
      GAMS=SIN(GAMA)
      GAMC=COS(GAMA)
      DO 6 IM=1,2
      IMS=K(IM)
      IF (IMS) 6,6,7
7     DO 5 IS=1,IMS
      THS=SIN(THATA(IM,IS))
      THC=COS(THATA(IM,IS))
      YS=Y*THS
      XO=1.0-X
      Z1=(YS*THS)/(2.0*XO)
      IF (1000.0*THC*THC.LT.Z1*Z1) GO TO 2
      R=CMPLX(0.0,(Z1+S(IM)*SQRT(Z1*Z1+THC*THC))/THC)
      Q=CMPLX(0.0,Y*THS*(1.0-N(IM,IS)**2)/(1.0-X))
      C=1.0/SQRT(1.0+CABS(R)**2)
      A(IM,IS,1)=C*(R*(Q*THS*GAMC-GAMS)-THC*GAMC)
      A(IM,IS,2)=C*(R*(Q*THS*GAMS+GAMC)-THC*GAMS)
      A(IM,IS,3)=C*(R*Q*THC+THS)
      B(IM,IS)=1.0+((CABS(R)*CABS(Q))**2)*C*C
      GO TO 5
2     GO TO (3,4),IM
3     A(IM,IS,1)=CMPLX(-THC*GAMC,0.0)
      A(IM,IS,2)=CMPLX(-THC*GAMS,0.0)
      A(IM,IS,3)=CMPLX(THS,0.0)
      B(IM,IS)=1.0
      GO TO 5
4     D=(X*YS)/(XO-YS*YS)
      E=XO*THC/YS
      F=-D*THS-E*THC/THS
      A(IM,IS,1)=CMPLX(F*GAMC,-GAMS)
      A(IM,IS,2)=CMPLX(F*GAMS,GAMC)
      A(IM,IS,3)=CMPLX(-D*THC+E,0.0)
      B(IM,IS)=1.0+(((2.0*Z1*D)**2)/(THC*THC+4.0*Z1*Z1))
5     CONTINUE
6     CONTINUE
      RETURN
      END
```

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13. ABSTRACT

This work treats the problem of scattering of electromagnetic waves from random electron density fluctuations in a cold magneto-plasma. A Green's function, asymptotic for large distances, is developed in terms of characteristic waves and used to solve two kinds of problems using the Born approximation. In the bistatic problem energy is beamed at a region containing random density fluctuations. An expression for the power scattered in a given direction is developed. In the radio star problem a wave propagates through an infinite slab. Expressions for the correlations of the in phase and quadriture phase components of the electric field along the unscattered ray path are obtained. There are two general conclusions (i) the solution can depend very strongly on the relationship among the incident ray, the scattered ray and the D.C. magnetic field, (ii) mode conversion is caused by those fluctuations whose correlation length is of the order of the wavelength.

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